Aportaciones Matemáticas
Comunicaciones 17 (1996) 57–64.

Pavement Stresses Due to Tire Impact

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This problem came from the Instituto Mexicano del Transporte, and was presented by Jose A. Romero-Navarrete.

Road surfaces wear continually under the effects of vehicular motions in an environment of changing temperature, humidity, etc. Regulatory agencies need to set limits on vehicular loads, tire pressures, etc., in order to mitigate the damage caused by the traveling stress footprints of vehicular traffic. In order to understand and quantify the relationship between damage caused and the parameters influencing the forces generated by a moving vehicle on a road surface, it is necessary to construct a model for a mechanical system of vehicle body, suspension springs, axle, wheel rim and tire, transmitting forces back and forward to the road surface.

The previous paragraph describes the broad problem presented to the workshop. In what follows we organize a simple mathematical model to represent the major components of the system, and we indicate how this model may be validated (or not) by tests and, if it is successful, how it can be used in a predictive capacity.

Vehicle/Suspension Spring/Tire System

We are interested in the changing stresses due to forward and vertical motions of a vehicle as they are transmitted to the road surface through the time-varying contact area (footprint) of the tire. We make the following simplications (see Fig. 1):

1. There is no vehicle forward motion.
2. The vehicle body is a point mass M.
3. The suspension spring is modeled as a linear spring, spring constant K_s.
4. The axle/rim is a point mass m ($m \ll M$, and m will be neglected in most of the analysis).

In modeling the tire we were guided by material from Johnson, [1], p. 277-279. An aircraft tire, which has little tread or sidewall structure, under load increases its contact with the road surface longitudinally and laterally. Reasonable results are obtained by taking the footprint as an ellipse with both axes changing under load. The footprint pressure is constant equal to the inflation pressure. In contrast, motor tires have a thick tread and have a complicated sidewall structure. The footprint is approximately rectangular and under load its length increases, but its width does not vary. The pressure across the footprint is not uniform, having peaks at the extremities of the width due to the sidewalls carrying load, and at the extremities of the length due to the bending stiffness of the tread.

It is instructive to consider how the pressure from the road surface is transmitted upwards to support the vehicle. There are two mechanisms at work:

(a) The inflated tire is under tension. Under load the tire sidewall between the rim and road surface is compressed, reducing the tensile force in the tire in this region. The net force on the rim is therefore upwards. (See Fig. 2a).

(b) The curvature of the tire sidewall between the rim and the road surface increases with load. This reduces the vertical component of the tire tension force on the lower part of the rim relative to the upper part of the rim, again resulting in a net upwards force. (See Fig. 2b).

The transmission of forces from the wheel rim through the tire to the road surface encompasses anisotropic shell theory. In order to make progress during the workshop we avoided detail at that level by making assumptions that

5. each sidewall element responds under compression with a restoring force proportional to the vertical displacement of the axle.

6. the number of sidewall elements responding to load is also proportional to the vertical displacement (this number corresponds to the length of the footprint).

The conjunction of (5) and (6) means that the net response of the tire is proportional to the square of its axle displacement, and the system consists of a linear spring in series with a non-linear one. This system is now set up and analyzed.

Mathematical Model

The vehicle mass M is at height $\overline{H}$, supported by the suspension spring, constant K_s, natural length L, which in turn is connected to the axle/rim, mass m, at height $\overline{h}$. This is supported by the tire modeled as an elastic spring of natural length l and of spring constant $K_t d$, where K_t is spring constant per unit length and d is a measure of the number of tire elements under compression, here taken as the length of the tire footprint. (See Fig. 1).

The equations of vertical motion for M and m are

$$M\frac{d^2\overline{H}}{dt^2} = K_s\left[L - (\overline{H} - \overline{h})\right] - Mg \tag{1}$$

$$m\frac{d^2\overline{h}}{dt^2} = -K_s\left[L - (\overline{H} - \overline{h})\right] - mg + K_t d(l - \overline{h}) \tag{2}$$

Under static conditions the accelerations are zero, and the variables $\overline{H}, \overline{h}, d$ have constant values $\overline{H}_0, \overline{h}_0, d_0$, respectively. Subtracting the static equations from the above removes the weight terms and yields

$$M\frac{d^2H}{dt^2} = -K_s(H - h) \tag{3}$$

$$m\frac{d^2h}{dt^2} = K_s(H - h) + K_t d(l - \overline{h}) - K_t d_0(l - \overline{h}_0) \tag{4}$$

where

$$H = \overline{H} - \overline{H}_0 \quad h = \overline{h} - \overline{h}_0$$

have been used, representing displacements from the static configuration.

We now invoke assumption (5) above and relate the footprint length to the vertical displacement of the tire. That is

$$d = c(l - \overline{h}), \quad d_0 = c(l - \overline{h}_0) \tag{5}$$

where c is a non-dimensional constant depending on parameters such as wheel radius, rim radius, tire pressure, sidewall structure, etc.

Remark Use of (5) in (2) indicates that the restoring force of a compressed tire is equal to $K_t c(l - \overline{h})^2$, the linear response of each sidewall element being effective over a footprint also taken linear in the displacement, resulting in the square law. This assumption can be tested directly for a given tire by compressing it between axle and road surface, and plotting force versus axle displacement. Values not according to the square law invalidate this model and the following analysis. However similar analysis can treat the measured law.

Substitution of (5) into (4) yields

$$m\frac{d^2h}{dt^2} = K_sH - (K_s + 2d_0K_t)h + cK_th^2. \tag{6}$$

Equations (3) and (6) now govern the motion of the vehicle/suspension spring/tire system, and the assumptions made on the tire response show up in the square term in (6). Since

$m \ll M$, and since there is considerable reduction in difficulty when the system reduces from fourth order to second order we shall take up the case $m = 0$.

Case $m = 0$: Second Order System
The equations become more accessible in non-dimensional form. Measuring time in units of ω^{-1} and measuring H, h in units of d_0 allow (3),(6) to be written as

$$\frac{d^2H}{dt^2} = h - H \quad \text{with} \quad H = (1 + 2\lambda)h - c\lambda h^2 \tag{7}$$

where

$$\omega^2 = K_s/M, \quad \lambda = d_0 K_t/K_s$$

Substituting for H in the differential equation in (7) yields

$$\frac{dg}{dt} = \frac{\lambda(2cg^2 + ch^2 - 2h)}{1 + 2\lambda - 2c\lambda h} \quad \text{where} \quad \frac{dh}{dt} = g \tag{8}$$

and from the ratio of these two equations we obtain the first-order differential equation

$$\frac{dg}{dh} = \frac{\lambda(2cg^2 + ch^2 - 2h)}{g(1 + 2\lambda - 2c\lambda h)} \tag{9}$$

Considerable quantitative information is available from the phase plane portrait of (9), that is the direction field of (9) in the $h - g$ plane. The critical points of (9) occur where the lines of infinite slope, that is $g = 0$ and $h = h_1 \equiv (1 + 2\lambda)/2c\lambda$ intersect those of zero slope. This latter is the ellipse.

$$(h - 1/c)^2 + 2g^2 = 1/c^2 \tag{10}$$

The points $P_0(h = 0 = g)$ and $P_1(h = 2/c,\ g = 0)$ are always critical points, and there are two others at the intersection of the ellipse (10) with the line $h = h_1$. The latter occurs when $h_1 < 2/c$, i.e. when $\lambda = d_0 K_t/K_s > 1/2$. There are then two cases $\lambda < 1/2 (> 1/2)$ of different behavior of the system, corresponding to a weak (strong) tire wall spring constant relative to suspension spring constant.(We do not take up the special case $\lambda = 1/2$.)

We take up these cases separately.

Case $m = 0$ and $\lambda < 1/2$: weak tire wall. Here there are two singular points: P_0, a center, and P_1, a saddle. A typical case ($c = 1, \lambda = 1/4, h_1 = 3$) is shown in Fig. 3. In a neighborhood of the origin, solution trajectories are closed curves representing system oscillations. At some critical magniture there is a limit cycle (not shown), outside of which the trajectories no longer close and proceed to infinity. These latter solutions invalidate model assumptions. Additionally, for the tire to remain in contact with the road, (i.e., it does not bounce) the requirement $d > 0$ must hold. Using (5), this is equivalent to requiring $h < \frac{1}{c}$. Thus the oscillatory solutions inside the limit cycle with $h < \frac{1}{c}$ seem to be reasonable, and the locus of the solution touching the line $h = \frac{1}{c}$ (and its dependence on system parameters) may be an indication of the magnitude limits beyond which the system breaks down.

Case $m = 0$ and $\lambda > 1/2$: strong tire wall. There are now four singular points: P_0 and P_1 are centers, and $P_2, P_3(h = h_1, g + \pm[h_1(1/c - h_1/2)]^{1/2})$ are saddles. A typical case $(c = 2/3, \lambda = 3/2, h_1 = 2)$ is shown in Fig. 4. In neighborhoods of both P_0 and P_1 solution trajectories are closed curves representing system oscillations. There are limit cycles round each center (not shown), and these limit cycles share a common boundary on $h = h_1$. Outside the limit cycles the trajectories proceed to infinity. The oscillations represented by the closed trajectories around P_0 are reasonable, and, again, the outer limit of these (touching the line $h = \frac{1}{c}$) may give information on critical system parameters. Solution trajectories outside this region invalidate model assumptions.

Stresses on Pavement

The force (superimposed on the weight) transmitted to the pavement during the system oscillation is $K_t d(l - h)$, or $K_t(l - h)$ per unit length of footprint, where $h(t)$ can be found from integrating (8) numerically for given initial conditions. In particular the maximum value (as a function of time) occurs at the $h < 0, g = 0$ ordinate in Figures 3, 4. This force is transmitted through the tire tread.The consequent distribution of pressure at the tread/pavement interface was estimated on the basis of assumptions not considered ideal (but the only ones available to us at the workshop): that the tire is solid and the pavement an elastic half-space. The pressuring of the former on the latter is known as the punch problem and has a well known solution. In our situation the pressure on the pavement is given by

$$p(y, t) = \frac{1}{\pi} K_t(l - h)(b^2 - y^2)^{-1/2} \tag{11}$$

where b is the half width of the tread, and y is the co-ordinate measured from the center-line of the tread. Shear and principal stresses in the pavement can be determined from this solution (see [1], pp.35-37). As indicated this punch solution is not considered appropriate for the tire tread, but (11) does have the feature of pressure peaks at the sidewall locations on the tire footprint.

Conclusion

A simple model of the vehicle/spring/tire system has been constructed and some of its properties evaluated. The model has enough structure to warrant comparison with the real system, and is flexible enough to allow changes/extensions if necessary. Qualitative information on the model from phase plane characteristics implied that critical values of system parameters could be obtained relatively easily from numerical solutions.

References

[1] Johnson, K. L., Contact Mechanics, Cambridge University Press, 1985.

Figures 3 and 4 were obtained using DEQSOLVE, and ODE solver from MATHLIB software package, Innosoft International Inc., Claremont, CA.

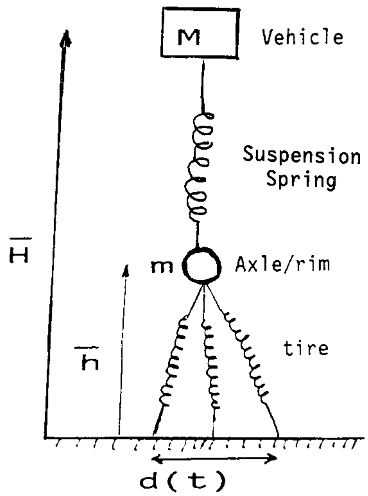

FIGURE 1

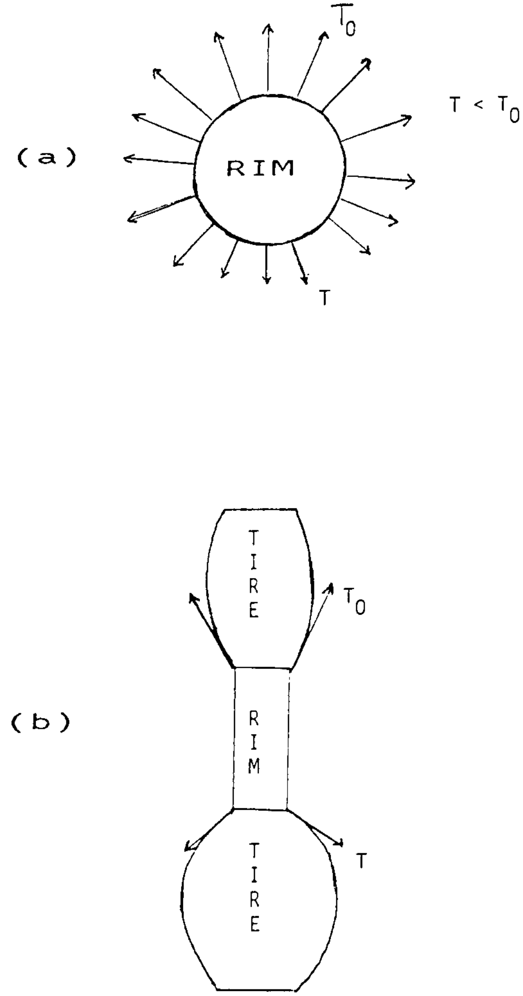

FIGURE 2

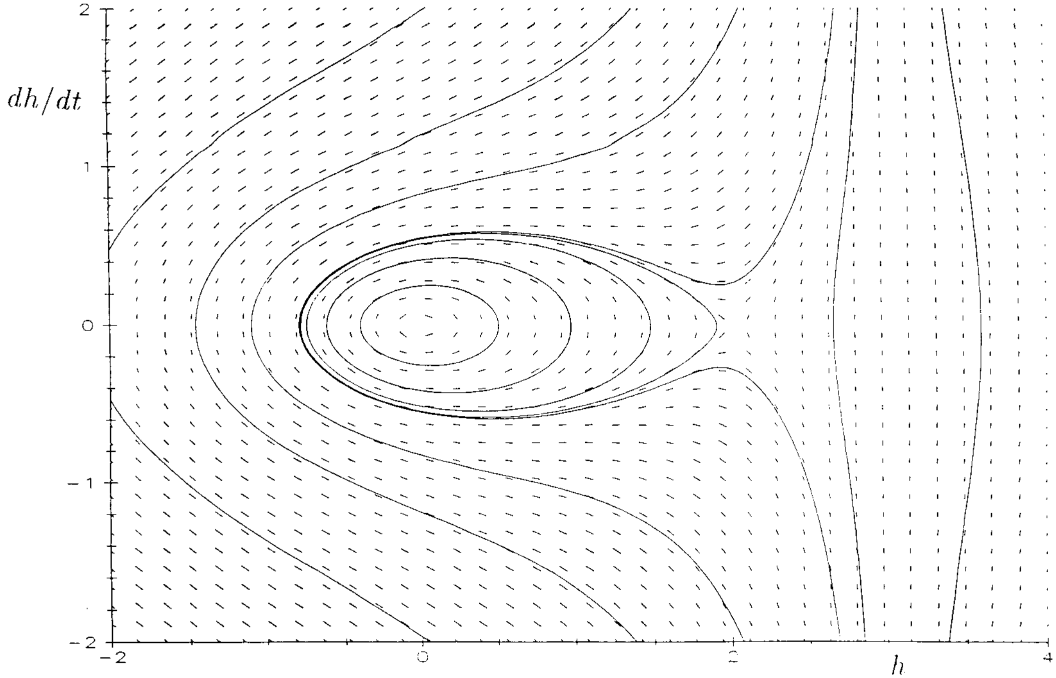

Figure 3: $c = 1, \quad \lambda = 1/4$

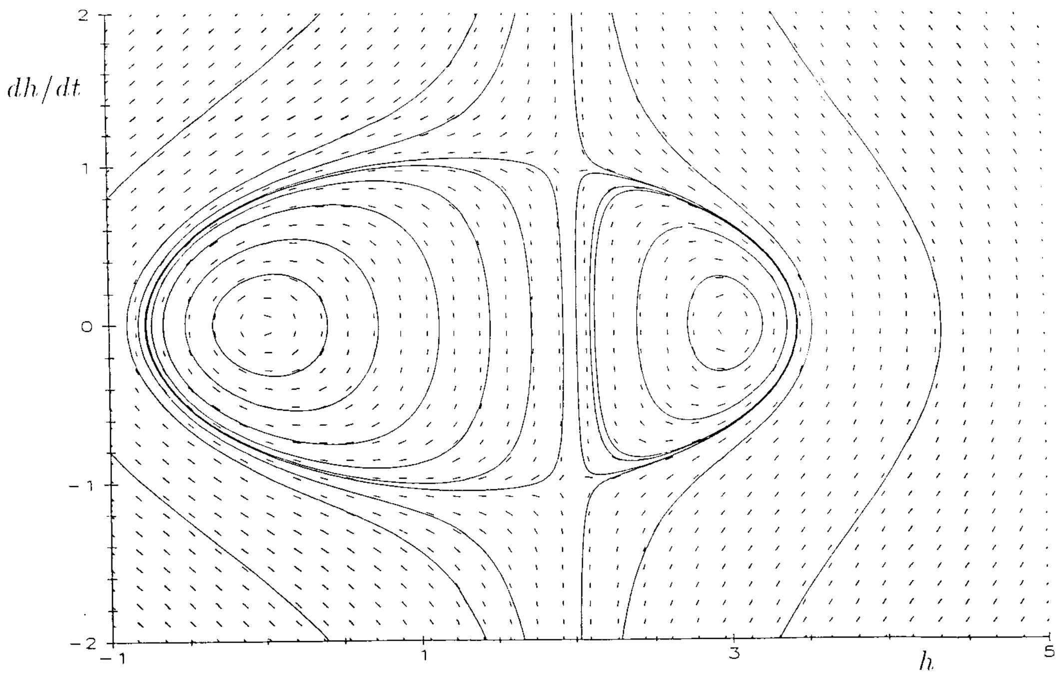

Figure 4: $c = 2/3, \quad \lambda = 3/2$