

# Testing and finding the generating functions $g$ of an option pricing mechanism through market data

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## Abstract

We study dynamic pricing mechanisms of financial derivatives. A typical model of such pricing mechanism is the so-called  $g$ -expectation defined by solutions of a backward stochastic differential equation with  $g$  as its generating function. Black-Scholes pricing model is a special linear case of this pricing mechanism. We are mainly concerned with two types of pricing mechanisms in an option market: the market pricing mechanism through which the market prices of options are produced, and the ask-bid pricing mechanism operated through the system of market makers. The later one is a typical nonlinear pricing mechanism. Data of prices produced by these two pricing mechanisms are usually quoted in an option market.

We introduce a criteria, i.e., the domination condition (A5) in (4.1) to test if a dynamic pricing mechanism under investigation is a  $g$ -pricing mechanism. This domination condition was statistically tested using CME data documents. The result of test is significantly positive. We also provide some useful characterizations of a pricing mechanism by its generating function.

**Keywords:** BSDE, dynamic pricing mechanism,  $g$ -expectation,  $g$ -martingale, dynamic risk measures.

## 1 Introduction

Dr. Faith Choy, on behalf of her team Reuters Hong Kong (Steven Cheung Deborah Au) proposed the following very interesting and important problem in option pricing: the well-known Black-Scholes formulae to price European options which was originated from the seminal papers of Black and Scholes (1973) and Merton (1973) with focuses on the following practically assumptions:

1. Create risk-free portfolios through dynamic hedging  
so that the portfolios earn risk-free interest rate;
2. Ensure an absence of arbitrage environment (Efficient markets);
3. Ensure no commission;
4. Ensure no interest rate change; and
5. Assume that stock returns follow a lognormal distribution.

They have proposed An alternative to Black-Scholes: pricing options with artificial intelligence and asked the following question: How to improve the non-linear transfer function  $g(\cdot)$  so that it can be effectively used to price options at low computation complexity? The study group has discussed this problem (Faith Choy, Benny Hon, Daniel Ho, Johnathan Wylie, Gerard Tronel, Shige Peng).

As it happen, our research group of Shandong University, collaborated with other research groups in mathematical finance, such as Ecole Polytechnique de Paris, Universite de Rennes I, ETH, Zurich,

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have struggled, for a quite longtime, a similar problem of finding the generating function  $g$  of a pricing mechanism of financial derivatives, including European call and put options, American options, Asian options, and some other exotic options.

The history of mathematical finance can be traced from Louis Bachelier's 1900 Thesis on option pricing. The theoretical and practical breakthrough of option pricing was Black-Scholes formula. But till then, the central problem of mathematical finance is still and, maybe always, the pricing mechanism of traders, market makers, small and large investors, and of a market.

We are agree with Reuters' opinion: in some sense a pricing mechanism of an financial institution is a blackbox. Its input is a derivative product and its output is the price of this , or the prices—since a market maker has the privilege to offer two prices: the ask price and the bid price. It is this dynamic blackbox of derivative pricing mechanism with, usually a huge quantity of input-output data that attract us to find if there is a generating function hidden behind us.

quantitatively describe the pricing mechanism of a market of derivatives is a very interesting problem. A model of dynamic pricing mechanism of derivatives is formulated (see (A1)–(A4) in the next section) to characterize this pricing behavior.

We are mainly concerned with two types of pricing mechanisms in an option market: the market pricing mechanism which outputs the trading prices of options and the bid–ask pricing mechanism operated according the system of market makers. We stress here that, in our point of view, the ask prices and the bid prices quoted in a market are determined by a single pricing mechanism. The difference of a ask price and the corresponding bid price, called bid–ask spread, reflects the nonlinearity of this mechanism. The data of prices of above mentioned two pricing systems is usually systematically quoted in the internet thus the models under our investigation can be statistically tested. We hope that our modelling can also be applied to describe the pricing mechanism of some other financial institutions.

The well-known Black–Scholes formula is a typical model of dynamic pricing mechanism of derivatives. It is a linear pricing mechanism. In fact, the prices produced by this mechanism is solved by a linear Backward Stochastic Differential Equation (BSDE). This means that the corresponding generating function  $g$  of the BSDE is a linear function. A nonlinear pricing mechanism by BSDE was originally proposed in [9]. In this report we show that each well-defined BSDE with a fixed generating function  $g$  forms a dynamic pricing mechanism, called  $g$ –expectation and that the behaviors of this mechanism are perfectly characterized by the behaviors of  $g$ . Several conditions of equivalence provided in this report will be very helpful to characterize and to find the generating function, or in some other circumstances, to regulate or to design a pricing mechanism.

A very interesting problem is how to design a test procedure to verify whether an existing pricing mechanism of derivatives is a  $g$ –expectation. We will present the following result: if a dynamic pricing mechanism is uniformly dominated by a  $g_\mu$ –expectation with a sufficiently large number  $\mu$  for the function  $g_\mu = \mu(|y| + |z|)$ , then it is a  $g$ –expectation. This domination inequality (4.1) has been applied as a testing criteria in our data analysis. The results strongly support that both the market pricing mechanism and the bid–ask pricing mechanism under our investigation can be modelled as  $g$ –expectations, and that the bid–ask prices are then produced by this single mechanism.

In this report we present the notion of  $g$ –pricing mechanism and show that, for each well–defined function  $g$  it satisfies the basic conditions (A1)–(A4) of a dynamic pricing mechanism of derivatives. We then show that, a dynamic pricing mechanism dominated by a  $g_\mu$ –expectation, i.e., (4.1) is satisfied, is a  $g$ –expectation. In Section 3, we will present some equivalent conditions to show that the behaviors of a  $g$ –expectation are perfectly reflected by its generating function  $g$ . We also provide some examples and explain how to statistically find the function  $g$  by testing the input–output data of prices.

We apply the crucial domination inequality (4.1) to test the market pricing mechanisms and the

bid-ask pricing mechanisms of S&P500 index future options and S&P500 index options, using data of parameter files provided by CME and CBOE. The result supports that they are  $g$ -expectations.

The main references of this report are [25] and [28].

## 2 A pricing mechanism of derivatives as an input-output blackbox

Let us consider an  $d$ -stock market with price  $S(t) = (S^1(t), \dots, S^d(t))$ . A derivative  $X$  underlying on the stock  $S$  with maturity  $T$  is a contract of which the value  $X$  is determined by the prices of  $S$  before  $T$ . A typical example is an European call option  $X = \max\{0, S^1(T) - k\}$  where the strike price  $k$  is fixed. In this case  $X$  depends only on  $S$  at the time  $T$ . An Asian option  $X = \frac{1}{T} \int_0^T S^1(t) dt$  depends on the whole prices of  $S^1$  before  $T$ . The central problem how, at a time  $t < T$ , an financial institute make a price of a derivative  $X$  with maturity  $T$ ?

A method quite different from the Black-Scholes's one is to formulate a pricing mechanism as an input-output box. This input-output system is significantly different from a traditional dynamic system e.g. a control system or a regulator in the sense that the input data, the option contract will be realized at its maturity time  $T$  whereas the output of this mechanism is the price of this option at the present time  $t < T$ . This means that the input data is realized after the output price. If we denote  $X_T$  to be the option contract whose data will be realized at the time  $T$ , the output of this pricing mechanism is the price of  $X_T$  at time  $t$ , denoted by  $\mathbb{O}_{t,T}[X_T]$ . However we make a basic and reasonable assumption that the input  $X_T$  depends on the prices of the underlying stocks  $(S_s)_{s \leq T}$ , before the maturity  $T$ , and the output  $\mathbb{O}_{t,T}[X_T]$  depends only on  $(S_s)_{s \leq t}$ .

Our main assumption on this dynamic blackbox  $\{\mathbb{O}_{t,T}[X_T]\}_{t \leq T}$  is as follows: for each  $s \leq t \leq T < \infty$ , derivative contracts  $X, \bar{X}$  depending on the price of  $S$  before  $t$ , the price of  $\mathbb{O}_{t,T}[X]$  (respectively the price of  $\mathbb{O}_{s,T}[X]$ ) depends on the price of  $S$  before  $t$  (respectively, before  $s$ ) and we have

$$(A1) \mathbb{O}_{t,T}[X] \geq \mathbb{O}_{t,T}[\bar{X}], \text{ if } X \geq \bar{X};$$

$$(A2) \mathbb{O}_{T,T}[X] = X;$$

$$(A3) \mathbb{O}_{s,t}[\mathbb{O}_{t,T}[X]] = \mathbb{O}_{s,T}[X]; \text{ for } s \leq t;$$

$$(A4) 1_A \mathbb{O}_{t,T}[X] = \mathbb{O}_{t,T}[1_A X], \text{ where } 1_A \text{ is valued on } 0 \text{ and } 1 \text{ and it depends on the prices of } S \text{ before } t.$$

**Remark 2.1.** (A1) and (A2) are economically obvious conditions for a rational derivative pricing mechanism. Condition (A3) means that, at the time  $s$ , the value  $\mathbb{O}_{t,T}^g[X]$  depends on the price of  $S$  before  $t$ , thus  $\mathbb{O}_{t,T}^g[X]$  can be regarded as a derivative contract with maturity  $t$ . The price of this derivative at  $s$  is  $\mathbb{O}_{s,t}^g[\mathbb{O}_{t,T}^g[X]]$ . This price must be the same as  $\mathbb{O}_{s,T}^g[X]$ .

**Remark 2.2.** The meaning of condition (A4) is that, since at time  $t$ , the agent knows already the value of  $I_A$  which either 1 or 0. When  $I_A$  is 1, then  $1_A X = X$  thus the price  $\mathbb{O}_{t,T}^g[1_A X]$  must be the same as  $\mathbb{O}_{t,T}^g[X]$ , otherwise  $1_A X = 0$ , so it worthies  $\mathbb{O}_{t,T}^g[1_A X] = \mathbb{O}_{t,T}^g[0] = 0 = 1_A \mathbb{O}_{t,T}^g[X]$ .

### 3 BSDE pricing mechanism model and Black-Scholes formula

Let us consider a market of financial derivatives in which the price  $(S(t))_{t \geq 0}$  of the underlying assets is driven by a  $d$ -dimensional Brownian motion  $(B_t)_{t \geq 0}$ . We assume that the past information  $\mathcal{F}_t^S$  of the price  $S$  before  $t$  depends on the values of  $B$  before  $t$ . A derivative  $X$  with maturity  $T$  is an  $\mathcal{F}_T$ -measurable and random value called maturity value.  $X$  is considered as an input. The output is the price  $Y_t$  of  $X$  at the time  $t < T$  of a given pricing mechanism.

Here we make a basic technique requirement that each process  $\eta_t$  is  $\mathcal{F}_t$ -adapted, namely, the value of  $\eta_t$  depends on the values of the Brownian motion  $B$  before  $t$ . Our BSDE pricing mechanism is to solve  $Y_t$  by the following backward stochastic differential equation (BSDE):

$$Y_t = X + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s. \quad (3.1)$$

Here  $(Y, Z)$  a pair of the  $\mathcal{F}_t$ -adapted processes to be solved,  $g$  is a given function

$$g : (\omega, t, y, z) \in \Omega \times [0, \infty) \times R \times R^d \rightarrow R.$$

We call  $g$  the generating function of the BSDE. It satisfies the following basic assumptions for each  $\forall y, \bar{y} \in R$  and  $z, \bar{z} \in R^d$ ,

$$|g(t, y, z) - g(T, \bar{y}, \bar{z})| \leq \mu(|y - \bar{y}| + |z - \bar{z}|). \quad (3.2)$$

It is important to consider the following special situation:

$$\begin{cases} \text{(a)} & g(\cdot, 0, 0) \equiv 0, \\ \text{(b)} & g(\cdot, y, 0) \equiv 0, \forall y \in R. \end{cases} \quad (3.3)$$

Obviously (b) implies (a). This BSDE (3.1) was introduced by Bismut [1] for the case where  $g$  is a linear function of  $(y, z)$ . [17, Pardoux-Peng, 1990] obtained the following basic result: for each  $X \in L^2(\mathcal{F}_T)$ , there exists a unique square-integrable adapted solution  $(Y, Z)$  of the BSDE (3.1). The following notion of  $g$ -expectations was introduced by [20, Peng 1997a] and [21, Peng 1997].

**Definition 3.1.** We denote by  $\mathbb{O}_{t,T}^g[X] := Y_t$ :

$$\mathbb{O}_{t,T}^g[\cdot] : L^2(\mathcal{F}_T) \rightarrow L^2(\mathcal{F}_t), \quad 0 \leq t \leq T < \infty. \quad (3.4)$$

$(\mathbb{O}_{t,T}^g[\cdot])_{0 \leq t \leq T < \infty}$  is called  $g$ -pricing mechanism.

As an example of BSDE pricing mechanism, we consider the following Black-Scholes pricing mechanism:

**Example 3.2. (Black-Scholes is a  $g$ -pricing mechanism)** Consider a financial market consisting of 2 underlying assets: one bond and one stock. We denote by  $S_0(t)$  the price of the bond and by  $S(t)$  the price of the stock at time  $t$ . We assume that  $S_0(t)$  satisfies an ordinary differential equation:  $dS_0(t) = r_t S_0(t) dt$ , and  $S(t)$  is the solution of the following stochastic differential equation (SDE) with 1-dimensional Brownian motion  $B$  (i.e.,  $d = 1$ ) as driven noise:

$$dS(t) = S(t)(b_t dt + \sigma_t dB_t), \quad S(0) = p.$$

Here  $r_t$  is the interest rate,  $b_t$  the rate of the expected return and  $\sigma_t$  the volatility of the stock at the time  $t$ .  $r_t, b_t, \sigma_t$  and  $\sigma^{-1}$  are assumed to be  $\mathcal{F}_t$ -measurable and uniformly bounded. Black and Scholes have solved the problem of the market pricing mechanism of an European call option  $X_{call} = \max\{0, S_T - k\}$  and put option  $X_{call} = \max\{0, k - S_T\}$ , where  $k$  is the strike price, under the assumption that  $r, b$  and  $\sigma$

are constant. Their main idea can be easily adapted to our slightly more general situation for a derivative  $X \in L^2(\mathcal{F}_T)$  with maturity  $T$ . Consider an investor with the following investment portfolio at a time  $t \leq T$ : he has  $n_0(t)$  bonds and  $n(t)$  stock, i.e., he invests  $n_0(t)S_0(t)$  in bond and  $\pi(t) = n(t)S(t)$  in the stock. We define by  $Y_t$  the investor's wealth invested in the market at time  $t$ :

$$Y_t = n_0(t)P_0(t) + n(t)P(t).$$

We make the so called "self-financing assumption":

$$dY_t = n_0(t)dS_0(t) + n(t)dS(t)$$

or

$$dY_t = [r_t Y_t + (b_t - r_t)\pi(t)]dt + \sigma_t \pi_t dB_t.$$

We denote  $g(t, y, z) := -r_t y - (b_t - r_t)\sigma_t^{-1}z$ . Then, by denoting  $Z_t = \sigma_t \pi(t)$ , the above equation is

$$-dY_t = g(t, Y_t, Z_t)dt - Z_t dB_t.$$

We observe that the above function  $g$  satisfies (3.2). It follows from the existence and uniqueness theorem of BSDE that for each derivative  $X \in L^2(\mathcal{F}_T)$ , there exists a unique adapted solution  $(Y, Z)$  with the terminal condition  $Y_T = X$ . This result of existence and uniqueness is economically meaningful: in order to replicate the derivative  $X$  at the maturity  $T$ , the investor needs and only needs to invest the  $Y_t$  at the present time  $t$  and then, during the time interval  $s \in [t, T]$ , to perform the portfolio strategy  $\pi(s) = \sigma_s^{-1}Z_s$ . Furthermore, by Comparison Theorem of BSDE, if he wants to replicate a derivative  $\bar{X}$  with the same maturity  $T$  which is bigger than  $X$  (i.e.,  $\bar{X} \geq X$  and  $P(\bar{X} \geq X) > 0$ ) then he must invest more than  $Y_t$  at the time  $t$ . This means there is no arbitrage opportunity. In this situation  $Y_t = \mathbb{O}_{t,T}^g[X]$  is called the Black-Scholes price, and  $(\mathbb{O}_{t,T}^g[\cdot])_{0 \leq t \leq T < \infty}$  is called the corresponding Black-Scholes pricing mechanism. We observe that the generating function  $g$  satisfies (a) of condition (3.3).

**Example 3.3.**  $X_{call} = \max\{0, S_T - K\}$ ,  $X_{put} = \max\{0, K - S_T\}$

$$\mathbb{O}_{t,T}^g[X_{call}] = S_t N(d_1) - K \exp(-r(T-t)) N(d_2)$$

$$\mathbb{O}_{t,T}^g[X_{put}] = K \exp(-r(T-t)) N(-d_2) - S_t N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-\frac{\eta^2}{2}) d\eta$$

**Example 3.4.** The following problem was considered in [5] and [9]: the investor is allowed to borrow money at time  $t$  at an interest rate  $R_t > r_t$ . The amount borrowed at time  $t$  is equal to  $(Y_t - \pi(t))^-$ . In this case the wealth process  $Y$  still satisfies BSDE:

$$-dY_t = g(t, Y_t, Z_t)dt - Z_t dW_t.$$

with  $g(t, y, z) := -r_t y - (b_t - r_t)\sigma_t^{-1}z + (R_t - r_t)(y - \sigma_t^{-1}z)^-$ . This derives a  $g$ -pricing mechanism with a sub-additive generating function  $g$ .

Similar equations appear in continuous trading with short sales constraints with different risk premium for long and short positions (cf. [15], [11] and [9]). In this case  $g(t, y, z) := -r_t y - (b_t - r_t)\sigma_t^{-1}z + k_t z^-$ . We observe that in each of the above three examples,  $g$  is sub-additive in  $(y, z)$ .

### 3.1 Numerical calculation of solutions of BSDEs

As the method of binormal tree in the option pricing model, our main idea of the calculation of BSDE is to replace the above Brownian motion  $B$  by a random walk. We refer to [29] for the details of our numerical approach. We make a standard time-partition of the interval  $[0, 1]$ :  $0 = t_0 < t_1 < \dots < t_n = 1$ ,  $\delta := t_j - t_{j-1} = \frac{1}{n}$ , for  $1 \leq j \leq n$ . Consider  $\{(\varepsilon_m)_{1 \leq m \leq n}\}$ , a Benoulli sequence, with  $\varepsilon_0 = 0$ , which are i.i.d. random variable satisfying

$$\varepsilon_m = \begin{cases} +1, & p = 0.5 \\ -1, & p = 0.5 \end{cases}.$$

Now we define the scaled random walk  $\{B^n\}$ , by setting  $B_0^n = 0$ ,

$$B_t^n = \sqrt{\delta} \sum_{m=0}^{\lfloor t/\delta \rfloor} \varepsilon_m, \quad 0 \leq t \leq T. \quad (3.5)$$

and denote  $B_j^n = B_{t_j}^n$ , i.e.,  $B_j^n = \sqrt{\delta} \sum_{m=1}^j \varepsilon_m$ . And we define the discrete filtration  $\mathcal{F}_j^n := \sigma\{\varepsilon_m; 0 \leq m \leq j\} = \sigma\{B_t^n; 0 \leq t \leq t_j\}$ , for  $1 \leq j \leq n$ .

Then on the small interval  $[t_j, t_{j+1}]$ , the equation

$$y_{t_j} = y_{t_{j+1}} + \int_{t_j}^{t_{j+1}} f(s, y_s, z_s) ds - \int_{t_j}^{t_{j+1}} z_s dB_s, \quad (3.6)$$

can be approximated by the discrete equation

$$y_j^n = y_{j+1}^n + f(t_j, y_j^n, z_j^n) \delta - z_j^n \sqrt{\delta} (B_{j+1}^n - B_j^n). \quad (3.7)$$

If  $f(t, y, z)$  satisfies the Lipschitz condition with constant  $k$ , for  $\delta < 1/k$ , there exists a unique couple  $(y_j^n, z_j^n)$  satisfying equation (3.7). In fact with  $B_{j+1}^n - B_j^n = \varepsilon_{j+1}$ , and  $E[\varepsilon_{j+1} | \mathcal{F}_j^n] = 0$ , we get immediately

$$z_j^n = \frac{1}{2\sqrt{\delta}} E[y_{j+1}^n \varepsilon_{j+1} | \mathcal{F}_j^n]. \quad (3.8)$$

Then taking conditional expectation on (3.7), it follows

$$y_j^n = E[y_{j+1}^n | \mathcal{F}_j^n] + f(t_j, y_j^n, z_j^n) \delta. \quad (3.9)$$

Consider the mapping  $\Theta(y) = y - f(t_j, y, z_j^n) \delta$ , from the Lipschitz property of  $f$ , we obtain

$$\langle \Theta(y) - \Theta(y'), y - y' \rangle \geq (1 - \delta k) |y - y'|^2 > 0,$$

which implies that the mapping  $\Theta(y)$  is a monotonic mapping. So there exists a unique value  $y$  s.t.  $\Theta(y) = E[y_{j+1}^n | \mathcal{F}_j^n]$  holds, i.e.  $y_j^n = \Theta^{-1}(E[y_{j+1}^n | \mathcal{F}_j^n])$ .

**Remark 3.5.** The existence of the solution of discrete BSDE only depends on the Lipschitz condition of  $f$  on  $y$ . In fact, if  $f$  does not depend  $y$ , we can easily get  $\Theta^{-1}(y) = y + f(t_j, z_j^n) \delta$ .

**Remark 3.6.** In general, if  $f$  nonlinearly depends on  $y$ , then  $\Theta^{-1}$  can not be solved explicitly, so sometimes we use  $(\bar{y}_j^n, \bar{z}_j^n)$ , where

$$\begin{aligned} \bar{y}_j^n &= E[y_{j+1}^n | \mathcal{F}_j^n] + f(t_j, E[y_{j+1}^n | \mathcal{F}_j^n], \bar{z}_j^n) \delta, \\ \bar{z}_j^n &= \frac{1}{2\sqrt{\delta}} E[y_{j+1}^n \varepsilon_{j+1} | \mathcal{F}_j^n], \end{aligned} \quad (3.10)$$

to approximate the solution for of  $\Theta(y) = E[y_{j+1}^n | \mathcal{F}_j^n]$ . And (3.10) is called the explicit scheme for BSDE, while (3.9) is called the implicit scheme for BSDE.

## 4 BSDE as a pricing mechanism of derivatives

The following result, obtained in [25]–Theorem 3.4, explains why this  $g$ -expectation is a good candidate to model a dynamic pricing mechanism of derivatives:

**Proposition 4.1.** *Let the generating function  $g$  satisfies (3.2) and (3.3)–(a). Then the above defined  $g$ -expectation  $\mathbb{O}^g[\cdot]$  is a dynamic pricing mechanism of derivatives, i.e., it satisfies, for each  $t \leq T < \infty$ ,  $X, \bar{X} \in L^2(\mathcal{F}_T)$ ,*

$$(A1) \quad \mathbb{O}_{t,T}^g[X] \geq \mathbb{O}_{t,T}^g[\bar{X}], \text{ a.s., if } X \geq \bar{X};$$

$$(A2) \quad \mathbb{O}_{T,T}^g[X] = X;$$

$$(A3) \quad \mathbb{O}_{s,t}^g[\mathbb{O}_{t,T}^g[X]] = \mathbb{O}_{s,T}^g[X]; \text{ for } s \leq t;$$

$$(A4) \quad 1_A \mathbb{O}_{t,T}^g[X] = \mathbb{O}_{t,T}^g[1_A X], \forall A \in \mathcal{F}_t,$$

where  $1_A$  is the indicator function of  $A$ , i.e.,  $1_A(\omega)$  equals to 1, when  $\omega \in A$  and 0 otherwise.

**Remark 4.2.** (A1) and (A2) are economically obvious conditions for a pricing mechanism. Condition (A3) means that, at the time  $s$ , the random value  $\mathbb{O}_{t,T}^g[X]$  can be regarded as a maturity value with maturity  $t$ . The price of this derivative at  $s$  is  $\mathbb{O}_{s,t}^g[\mathbb{O}_{t,T}^g[X]]$ . It must be the same as the price  $\mathbb{O}_{s,T}^g[X]$  of  $X$  at  $s$ .

**Remark 4.3.** The meaning of condition (A4) is that, since at time  $t$ , the agent knows the value of whether  $1_A$  is 1 or 0. When  $1_A$  is 1, then the price  $\mathbb{O}_{t,T}^g[1_A X]$  of  $1_A X$  must be the same as  $\mathbb{O}_{t,T}^g[X]$ , otherwise  $1_A X = 0$ , so it worthies 0.

From the above results we see that  $\mathbb{O}^g$  is a good candidate to be a dynamic pricing mechanism. The following result provides a criteria to test if a dynamic pricing mechanism is a  $g$ -expectation. The proof can be found in [27].

**Definition 4.4.** A system of mappings  $(\mathbb{O}_{t,T}[\cdot])_{0 \leq t \leq T < \infty}$

$$\mathbb{O}_{t,T}[X] : X \in L^2(\mathcal{F}_T) \mapsto L^2(\mathcal{F}_t)$$

is called a dynamic pricing mechanism of derivatives if it satisfies (A1)–(A4) (with  $\mathbb{O}[\cdot]$  in the place of  $\mathbb{O}^g[\cdot]$ ).

**Theorem 4.5.** Let  $(\mathbb{O}_{t,T}[\cdot])_{0 \leq t \leq T < \infty}$  be a dynamic pricing mechanism. If there exists a sufficiently large constant  $\mu > 0$ , such that the following domination criteria is satisfied

(A5):

$$\mathbb{O}_{t,T}[X] - \mathbb{O}_{t,T}[\bar{X}] \leq \mathbb{O}_{t,T}^{g_\mu}[X - \bar{X}]. \quad (4.1)$$

$\mathbb{O}^{g_\mu}$  is a  $g$ -expectation with the generating function  $g_\mu$  defined by

$$g_\mu(y, z) := \mu|y| + \mu|z|, (y, z) \in \mathbb{R} \times \mathbb{R}^d. \quad (4.2)$$

Then there exists a unique generating function  $g(\omega, t, y, z)$  satisfying (3.2) and (3.3)–(a) such that, for each  $t \leq T$  and for each derivative  $X \in L^2(\mathcal{F}_T)$ , we have

$$\mathbb{O}_{t,T}[X] = \mathbb{O}_{t,T}^g[X], \quad (4.3)$$

namely  $\mathbb{O}$  is a  $g$ -expectation.

**Remark 4.6.** This theorem also implies that, for a generating function  $g$  satisfying (3.2) and (3.3)–(a), the corresponding  $g$ -expectation  $\mathbb{O}^g$  is also dominated by  $\mathbb{O}^{g_\mu}$ , i.e., (A5) is satisfied. This can be also directly proved by using the comparison theorem of BSDE.

**Remark 4.7.** It turns out that the domination condition (4.1) becomes a crucial criteria to test whether a dynamic pricing mechanism of derivatives is a  $g$ -expectation. We provide a test in Appendix 4.2 to use market data to check the inequality (4.1).

**Remark 4.8.** This deep result has non-trivially generalized the main result of [4], theoretically and practically, where a special case  $g = g(t, z)$  with  $g(s, 0) \equiv 0$  is considered. The  $g$ -expectation originally introduced in [21] corresponds such situation of “zero interest rate”. (cf. [25]).

## 5 Testing the criteria (A5) by market data

With Chen L. and Sun P. of our research group, we proceed a data test for the criteria (A5), i.e., the domination inequality (4.1), to check if a specific pricing mechanism is a  $g$ -expectation, or  $g$ -pricing mechanism  $\mathbb{O}^g$ .

We have firstly tested the CME (Chicago Mercantile Exchange)’s market pricing mechanism of derivatives by taking the daily closing prices of options with S&P500 index futures as the underlying asset. The data is obtained from parameter files published from CME’s ftp-webset, named cmeMMDDs.par (MM for month, DD for day) of call and put prices, from 05 January 2000 to November 2003, of totally 960 trading days. The corresponding S&P500 future’s prices is obtained from the parameter files of SPAN (Standard Portfolio Analysis of Risk) system downloaded from CME’s ftp site.

We denote by  $X_T^i = (S_T - k_i)^+$  (resp.  $Y_T^i = (S_T - k_i)^-$ ), the market maturity value of the call (resp. put) option with maturity  $T$  and strike price  $k_i$ . The corresponding values of the short positions are  $-X_T^i$  and  $-Y_T^i$ . We denote the market price of the corresponding prices of options at time  $t < T$  by  $\mathbb{O}_{t,T}^m[X_T^i]$ ,  $\mathbb{O}_{t,T}^m[Y_T^i]$ ,  $\mathbb{O}_{t,T}^m[-X_T^i]$  and  $\mathbb{O}_{t,T}^m[-Y_T^i]$  respectively. The inequalities we need to put to the test are, according to (4.1), in the following different combinations, with different  $(t, T)$  and different strike prices

$$\left\{ \begin{array}{ll} \text{Call-Call:} & \mathbb{O}_{t,T}^m[X_T^i] - \mathbb{O}_{t,T}^m[X_T^j] \leq \mathbb{O}_{t,T}^{g\mu}[X_T^i - X_T^j] \\ \text{Put-Put:} & \mathbb{O}_{t,T}^m[Y_T^i] - \mathbb{O}_{t,T}^m[Y_T^j] \leq \mathbb{O}_{t,T}^{g\mu}[Y_T^i - Y_T^j] \\ \text{Call-Put:} & \mathbb{O}_{t,T}^m[X_T^i] - \mathbb{O}_{t,T}^m[Y_T^j] \leq \mathbb{O}_{t,T}^{g\mu}[X_T^i - Y_T^j] \\ \text{Put-Call:} & \mathbb{O}_{t,T}^m[Y_T^i] - \mathbb{O}_{t,T}^m[X_T^j] \leq \mathbb{O}_{t,T}^{g\mu}[Y_T^i - X_T^j] \end{array} \right. \quad (5.1)$$

and

$$\left\{ \begin{array}{ll} \text{Call-ShortCall:} & \mathbb{O}_{t,T}^m[X_T^i] - \mathbb{O}_{t,T}^m[-X_T^j] \leq \mathbb{O}_{t,T}^{g\mu}[X_T^i + X_T^j] \\ \text{Put-ShortPut:} & \mathbb{O}_{t,T}^m[Y_T^i] - \mathbb{O}_{t,T}^m[-Y_T^j] \leq \mathbb{O}_{t,T}^{g\mu}[Y_T^i + Y_T^j] \\ \text{Call-ShortPut:} & \mathbb{O}_{t,T}^m[X_T^i] - \mathbb{O}_{t,T}^m[-Y_T^j] \leq \mathbb{O}_{t,T}^{g\mu}[X_T^i + Y_T^j] \end{array} \right. \quad (5.2)$$

In the above inequalities the data of the left hand sides is the market prices of options taken from CME parameter files. In our testing the transaction cost is neglected, i.e., we assume that  $\mathbb{O}_{t,T}^m[-X] = -\mathbb{O}_{t,T}^m[X]$ . The right hand sides is the corresponding values of  $g_\mu$ -expectations. We fix  $\mu = 25$  uniformly for all tested inequalities. We have calculated all these values on the right hand side by using standard binomial tree algorithm of BSDE. Here an improved version of the algorithms of BSDE proposed Peng and Xu [2005] has been applied to solve the following 1-dimensional BSDE:

$$y_t = X + \int_t^T \mu(|y_s| + |z_s|)ds - \int_t^T z_s dB_s \quad (5.3)$$

with different terminal values  $y_T = X_T^i - X_T^j, Y_T^i - Y_T^j, X_T^i - Y_T^j, Y_T^i - X_T^j, X_T^i + X_T^j, Y_T^i + Y_T^j, X_T^i + Y_T^j$ , respectively. The closing prices of S&P500 futures options of 69 trading days from year 2000 to 2003 have been put in the test. With the above mentioned combinations, we have tested a total number

of 6,200,828 inequalities of (5.1) and (5.2). This means that our BSDE (5.3) have been calculated 6,200,828 times. A very positive result was obtained: among the totally 6,200,828 tested inequalities, only 17 are against the criteria (5.1). Among those 12 cases of violations, 5 are singular situation since they themselves all violate Axiomatic monotonicity condition (A1). 5 cases are all from the same file cme0701s.par, 2003, Put–Put. They are all the following singular cases:

$$\mathbb{O}_{t,T}^m[(S_T - k_i)^-] > \mathbb{O}_{t,T}^m[(S_T - k_j)^-], \text{ for } k_i > k_j.$$

The other 12 violations are the cases where the time  $T - t$  is too short (less than 2 days).

Since we have not found available data of bid-ask prices of the above options from CME, we then have tested the bid-ask pricing mechanism of S&P500 index options operated by the system of market makers of CBOE. The data source is from Yahoo's finance quotes of the option prices from 07 December to 08 May 2006. We have collected the prices of 5,000 time points, i.e., 5,000 different  $t$  among 100 trading days. We denote this pricing mechanism by  $\mathbb{O}_{t,T}^{mm}[X]$  for the ask price of an option  $X$ . According to our point of view the bid price of the same  $X$  is  $-\mathbb{O}_{t,T}^{mm}[-X]$  and thus the bid–ask spread is  $\mathbb{O}_{t,T}^{mm}[X] + \mathbb{O}_{t,T}^{mm}[-X]$ . We have tested a total number of 589,360 inequalities of (5.1) and (5.2), with  $\mathbb{O}^{mm}$  in the place of  $\mathbb{O}^m$ . Only 1 case of violation appears.

We will report these test results in details in our forthcoming paper. [3].

**Markovian pricing mechanisms** We limit ourselves to consider, for each fixed maturity  $T$ , the derivatives  $X$  depending only on the price  $S_T$ , i.e.,  $X$  is a path independent derivative.  $X$  is then in the class of

$$X = \Phi(S_T) \text{ with } \Phi \in L^2(S_T)$$

where  $L^2(S_T)$  denotes the collection of all real functions  $\Phi$  defined on  $R^n$  such that  $\Phi(S_T) \in L^2(\mathcal{F}_T)$ . A dynamic pricing mechanism  $\mathbb{O}$  is called a Markovian pricing mechanism if for each  $0 \leq t \leq T < \infty$  and  $\Phi \in L^2(S_T)$  there exists  $\phi \in L^2(S_t)$  such that  $\mathbb{O}_{t,T}[\Phi(S_T)] = \phi(S_t)$ . In other words, *the price of a path-independent option by a Markovian pricing mechanism is still path-independent.*

**Example 5.1.** We consider a situation where the underlying price  $S$  is a diffusion process:

$$dS_t = b(S_t)dt + \Lambda(S_t)dB_t, \quad S_0 = s_0 \in R^n.$$

where  $b$  and  $\Lambda$  are given Lipschitz functions of  $R^n$  valued on  $R^n$  and  $R^{n \times d}$  respectively. If a generating function  $g$  has the following form:

$$g(t, y, z) = f(S_t, y, z),$$

where  $f$  is a Lipschitz function of  $(s, y, z) \in R^n \times R \times R^d$ . By the nonlinear Feynman–Kac formula introduced in [19, Peng 1991] and developed in [18, Pardoux–Peng 1992], for each option  $X = \Phi(S_T)$  with smooth function  $\Phi$  the price of the related  $g$ -expectation is

$$\mathbb{O}_{t,T}^g[\Phi(S_T)] = u(t, S_t)$$

where  $u : R_+ \times R^n \mapsto R$  is the (viscosity) solution of the following PDE defined on  $(t, s) \in [0, T] \times R^n$ :

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{i,j=1}^n (\Lambda(s)\Lambda^T(s))_{ij} \frac{\partial^2 u}{\partial s_i \partial s_j} + \sum_{i=1}^n b_i(s) \frac{\partial u}{\partial s_i} + f(s, u, \Lambda^T(s)\nabla u) = 0$$

with terminal condition  $u(T, s) = \Phi(s)$ . If  $S_t$  is a 1-dimensional geometric Brownian motion, i.e.,  $\Lambda(s) = \sigma s$  and  $b(s) = \mu s$ , then the above PDE becomes

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + \mu s \frac{\partial u}{\partial s} + f(s, u, \sigma s \frac{\partial u}{\partial s}) = 0.$$

The Black–Scholes formula corresponds to the case  $f = -ry - (\mu - r)\sigma^{-1}z$ . We then have

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} - ru + rs \frac{\partial u}{\partial s} = 0, \quad u(T, s) = \Phi(s).$$

## 6 Characterization of $g$ -pricing mechanism by its generating function $g$

For a pricing mechanism, it is important to distinguish the selling price and buying price of a same pricing mechanism, corresponding to the ask price and the bid price if the mechanism under investigation is generated through the system of market makers of an option market (cf. [12] Sec. 6.5 and [16]). If  $\mathbb{O}_{t,T}[X]$  is the ask price at the time  $t$  of a derivative  $X$  with maturity  $T$ , then the bid price must be  $-\mathbb{O}_{t,T}[-X]$  and we have, in general,

$$\mathbb{O}_{t,T}[X] > -\mathbb{O}_{t,T}[-X].$$

Here we stress our point of view that, in fact, the ask price and bid price are produced by a single mechanism, called bid–ask pricing mechanism of market makers. Our result of data analysis to test the criteria (A5) of the domination condition (4.1) strongly supports this point of view. Moreover, this analysis also supports our point of view that, for a well–developed market, there exist a function  $g$  satisfying Lipschitz condition (3.2) such that the corresponding ask–bid pricing mechanism is modeled by the  $g$ -expectation  $\mathbb{O}^g[\cdot]$ .

A rational dynamic pricing mechanism also possesses some other important properties, such as convexity, sub-additivity. See [9], [22], [30], [13], [14] among many others. We will see that the generating function  $g$  perfectly reflects the behavior of  $\mathbb{O}^g$ . This may play an important role to statistically find  $g$  by using the corresponding data of prices. In the following we provide several theoretical results with proofs given in Appendix. This problem was treated also by [30], [13] and [14].

**Proposition 6.1.** *Let  $g, \bar{g} : (\omega, t, y, z) \in \Omega \times [0, \infty) \times R \times R^d \rightarrow R$  be two generating functions satisfying (3.2). Then the following two conditions are equivalent:*

- (i)  $g(\omega, t, y, z) \geq \bar{g}(\omega, t, y, z), \forall (y, z) \in R \times R^d, dP \times dt$  a.s.
- (ii) *The corresponding  $g$ -pricing mechanisms  $\mathbb{O}^g[\cdot]$  and  $\mathbb{O}^{\bar{g}}[\cdot]$  satisfy*

$$\mathbb{O}_{t,T}^g[X] \geq \mathbb{O}_{t,T}^{\bar{g}}[X], \forall 0 \leq t \leq T < \infty, \forall X \in L^2(\mathcal{F}_T).$$

*In particular,  $\mathbb{O}^g[X] \equiv \mathbb{O}^{\bar{g}}[X]$  if and only if  $g \equiv \bar{g}$ .*

**Corollary 6.2.** *The following two conditions are equivalent:*

- (i) *The generating function  $g$  satisfies, for each  $(y, z) \in R \times R^d$ ,*

$$g(t, y, z) \geq -g(t, -y, -z), \text{ a.e., a.s.,}$$

- (ii)  $\mathbb{O}_{t,T}^g[\cdot] : L^2(\mathcal{F}_T) \mapsto L^2(\mathcal{F}_t)$  *satisfies, for each  $0 \leq t \leq T$ ,*

**Proposition 6.3.** *The following two conditions are equivalent:*

- (i) *The generating function  $g = g(t, y, z)$  is convex (resp. concave) in  $(y, z)$ , i.e., for each  $(y, z)$  and  $(\bar{y}, \bar{z})$  in  $R \times R^d$  and for a.e.  $t \in [0, T]$ ,*

$$\begin{aligned} g(t, \alpha y + (1 - \alpha)\bar{y}, \alpha z + (1 - \alpha)\bar{z}) &\leq \alpha g(t, y, z) + (1 - \alpha)g(t, \bar{y}, \bar{z}), \text{ a.s.} \\ (\text{resp. } &\geq \alpha g(t, y, z) + (1 - \alpha)g(t, \bar{y}, \bar{z}), \text{ a.s.}) \end{aligned}$$

- (ii) *The corresponding pricing mechanism  $(\mathbb{O}_{t,T}^g[\cdot])_{0 \leq t \leq T < \infty}$  is convex (resp. concave), i.e., for each fixed  $\alpha \in [0, 1]$ , we have*

$$\begin{aligned} \mathbb{O}_{t,T}^g[\alpha X + (1 - \alpha)\bar{X}] &\leq \alpha \mathbb{O}_{t,T}^g[X] + (1 - \alpha)\mathbb{O}_{t,T}^g[\bar{X}], \text{ a.s.} \\ (\text{resp. } &\geq \alpha \mathbb{O}_{t,T}^g[X] + (1 - \alpha)\mathbb{O}_{t,T}^g[\bar{X}], \text{ a.s.}) \end{aligned} \tag{6.1}$$

*for each  $t \leq T$ , and  $X, \bar{X} \in L^2(\mathcal{F}_T)$ .*

**Proposition 6.4.** *The following two conditions are equivalent:*

(i) *The generating function  $g$  is positively homogenous in  $(y, z) \in R \times R^d$ , i.e.,*

$$g(t, \lambda y, \lambda z) = \lambda g(t, y, z), \text{ a.e., a.s.,}$$

(ii) *The corresponding pricing mechanism  $\mathbb{O}_{t,T}^g[\cdot] : L^2(\mathcal{F}_T) \mapsto L^2(\mathcal{F}_t)$  is positively homogenous: for each  $0 \leq t \leq T$ , i.e.,  $\mathbb{O}_{t,T}^g[\lambda X] = \lambda \mathbb{O}_{t,T}^g[X]$ , for each  $\lambda \geq 0$  and  $X \in L^2(\mathcal{F}_T)$ .*

From the above two propositions we immediately have

**Corollary 6.5.** *The following two conditions are equivalent:*

(i) *The generating function  $g$  is sub-additive: for each  $(y, z), (\bar{y}, \bar{z}) \in R \times R^d$ ,*

$$g(\omega, t, y + \bar{y}, z + \bar{z}) \leq g(\omega, t, y, z) + g(\omega, t, \bar{y}, \bar{z}), \text{ dt} \times \text{dP, a.s.,}$$

(ii) *The corresponding pricing mechanism  $\mathbb{O}_{t,T}^g[\cdot] : L^2(\mathcal{F}_T) \mapsto L^2(\mathcal{F}_t)$  is sub-additive: for each  $0 \leq t \leq T$  and  $X, \bar{X} \in L^2(\mathcal{F}_T)$*

$$\mathbb{O}_{t,T}^g[X + \bar{X}] \leq \mathbb{O}_{t,T}^g[X] + \mathbb{O}_{t,T}^g[\bar{X}].$$

**Proposition 6.6.** *The generating function  $g$  is independent of  $y$  if and only if the corresponding  $g$ -expectation satisfies the following “cash translatability” property: for each  $t \leq T$ ,*

$$\mathbb{O}_{t,T}^g[X + \eta] = \mathbb{O}_{t,T}^g[X] + \eta, \text{ a.s., for each } X \in L^2(\mathcal{F}_T), \eta \in L^2(\mathcal{F}_t).$$

We consider the following self-financing condition:

$$\mathbb{O}_{t,T}^g[0] \equiv 0, \forall 0 \leq t \leq T.$$

**Proposition 6.7.**  $\mathbb{O}^g[\cdot]$  *satisfies the self-financing condition if and only if its generating function  $g$  satisfies (3.3)–(a).*

*Proof.* The “if” part is obvious. The “only if part”:  $Y_t := \mathbb{O}_{t,T}^g[0] \equiv 0$ , implies

$$Y_t \equiv 0 \equiv 0 + \int_t^T g(s, 0, Z_s) ds - \int_t^T Z_s dB_s, \text{ } t \in [0, T].$$

Thus  $Z_t \equiv 0$  and then  $g(t, 0, Z_t) = g(t, 0, 0) \equiv 0$ . □

“Zero-interest rate” condition:

$$\mathbb{O}_{t,T}^g[\eta] = \eta, \forall 0 \leq t \leq T < \infty, \eta \in L^2(\mathcal{F}_t).$$

**Proposition 6.8.**  $\mathbb{O}^g[\cdot]$  *satisfies the zero-interest rate condition if and only if its generating function  $g$  satisfies (3.3)–(b).*

**Proposition 6.9.** *The following condition are equivalent:*

(i) *For each  $0 \leq t \leq T$  and  $X \in L^2(\mathcal{F}_T^t)$ , the  $g$ -pricing mechanism  $\mathbb{O}_{t,T}^g[X]$  is a deterministic number;*  
(ii) *The corresponding pricing generating function  $g$  is a deterministic function of  $(t, y, z) \in [0, T] \times R \times R^d$ .*

The proof is similar as the others. We omit it.

**Example 6.10.** An interesting problem is: if we know that a pricing mechanism under our investigation is a  $g$ -expectation  $\mathbb{O}^g$ , how to find the generating function  $g$ ? If we limited ourselves to only take data of prices quoted by markets, this is still an open problem. We now consider a case of “toy model” where  $g$  depends only on  $z$ , i.e.,  $g = g(z) : \mathbf{R}^d \rightarrow \mathbf{R}$ . We will find such  $g$  by the following testing method. Let  $\bar{z} \in \mathbf{R}^d$  be given. We denote  $Y_s := \mathbb{O}_{s,T}^g[\bar{z} \cdot (B_T - B_t)]$ ,  $s \in [t, T]$ , where  $t$  is the present time. It is the solution of the following BSDE

$$Y_s = \bar{z} \cdot (B_T - B_t) + \int_s^T g(Z_u) du - \int_s^T Z_u dB_u, \quad s \in [t, T].$$

It is seen that the solution is  $Y_s = \bar{z} \cdot (B_s - B_t) + \int_s^T g(\bar{z}) ds$ ,  $Z_s \equiv \bar{z}$ . Thus

$$\mathbb{O}_{t,T}^g[\bar{z} \cdot (B_T - B_t)] = Y_t = g(\bar{z})(T - t),$$

or

$$g(\bar{z}) = (T - t)^{-1} \mathbb{O}_{t,T}^g[\bar{z} \cdot (B_T - B_t)]. \quad (6.2)$$

Thus the function  $g$  can be tested as follows: at the present time  $t$ : if the valuation  $\mathbb{O}_{t,T}^g[\bar{z} \cdot (B_T - B_t)]$  of (a toy model of) derivative  $\bar{z} \cdot (B_T - B_t)$  is obtained, then  $g(\bar{z})$  is explicitly given by (6.2). We observe that, in the case where  $S$  is a geometric Brownian motion,  $B_T - B_t$  can be expressed as a function of  $S_T/S_t$ . But this cannot be applied to a general situation.

**Remark 6.11.** The above test is also applied for the case  $g : [0, \infty) \times \mathbf{R}^d \rightarrow \mathbf{R}$ , or for a more general situation  $g = \gamma y + g_0(t, z)$ .

An interesting problem is, in general, how to find the generating function  $g$  by a testing of the input–output behavior of  $\mathbb{O}^g[\cdot]$ ? Let  $b : \mathbf{R}^n \rightarrow \mathbf{R}^n$ ,  $\bar{\sigma} : \mathbf{R}^n \rightarrow \mathbf{R}^{n \times d}$  be two Lipschitz functions. For each  $(t, x) \in \mathbf{R}_+ \times \mathbf{R}^n$ , we consider the SDE of the form

$$X_s^{t,x} = x + \int_t^s b(X_s^{t,x}) ds + \int_t^s \sigma(X_s^{t,x}) dB_s, \quad s \geq t.$$

This SDE is regarded as the equation of the price of the underlying stock. The following result was obtained in Proposition 2.3 of [2].

**Proposition 6.12.** We assume that the generating function  $g$  satisfies (3.2). We also assume that, for each fixed  $(y, z)$ ,  $g(\cdot, y, z) \in D_{\mathcal{F}}^2(0, T)$  (the space of all  $\mathcal{F}_t$ -adapted processes with RCLL paths). Then for each  $(t, x, p, y) \in [0, \infty) \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ , we have

$$L^2\text{-}\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [\mathbb{O}_{t,t+\epsilon}^g[y + p \cdot (X_{t+\epsilon}^{t,x} - x)] - y] = g(t, y, \sigma^T(x)p) + p \cdot b(x).$$

## 7 Conclusion

We have our huge member of tests, using market data and numerical BSDE calculation technique, strongly support that a the market ask and bid pricing mechanism of CME options on S&P500 is a  $g$ -pricing mechanism. We have also proposed different types of generating functions  $g$  (convex function, concave sublinear function, superlinear function) to be good candidates of different pricing mechanisms for sellers, buyers, large investors and small investors. But to solve the problem how to test an specific financial company’s pricing mechanism, we need more specific data which is, in general, a high secret.

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### Non-linear g-expectation

$$y_t = \xi + \int_t^T g(s, y_s, z_s) ds - \int_t^T z_s d\bar{B}_s$$

$$t \in [0, T], \quad T = 1$$

$$\xi_g[\xi] = y_0$$

input  $g(t,y,z)$ :

$\xi = \Phi(B(1))$ , input  $\Phi(x)$ :

calculate

progress

brownian motion

solution (y, z)

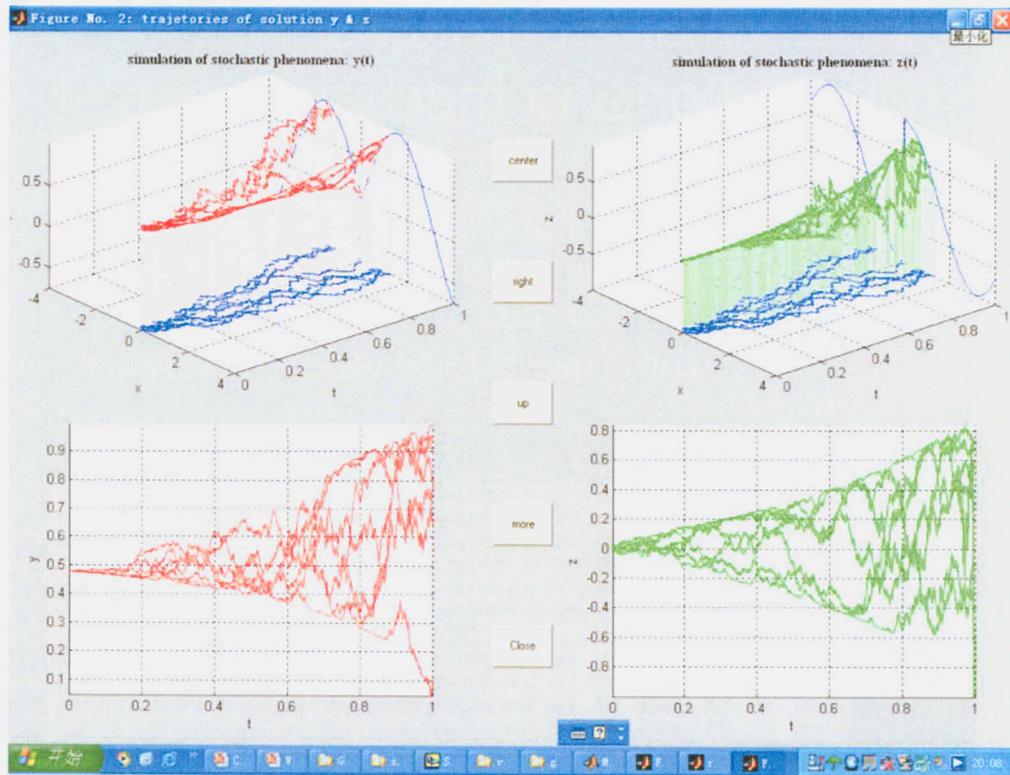
B.M & solution y

distribution

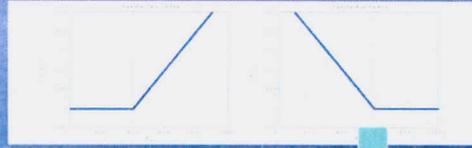
Close

realized by XU Mingyu, directed by PENG Shige  
in Institute of Mathematics, Shandong University

To calculate the output  $y_t$ , we input the derivative  $\xi$   
toolbox of g-pricing calculations



- X-Y: Vanilla Option  
different combinations



## CME's market maker's ask-bid pricing mechanism's test

- (CBOE)S&P500-index option.
- Data source: Yahoo(2004-12-07—200-05-08日  
101days time points
- 3,692,580's inequality to be tested: no violation happens

SZPDQ	29.20	+8.70	4.00	30.10	31.60	5,351.00	1,285.00	SZPQ	3.10	-1.00	1,322.00	2.75	3.40	12,182.00
SZPDR	28.20	+4.50	3.00	25.90	27.40	13,031.00	1,290.00	SZPR	3.80	-1.30	3,675.00	3.70	3.90	28,198.00
SZPDS	24.10	+5.10	45.00	21.90	23.40	6,420.00	1,295.00	SZPPS	4.70	-1.70	4,166.00	4.70	5.10	28,184.00
SXYDT	19.10	+2.90	1,691.00	18.50	19.50	58,185.00	1,300.00	SXYPT	6.10	-1.80	3,164.00	5.90	6.20	46,912.00
SXYDA	16.50	+3.30	596.00	14.90	16.10	14,188.00	1,305.00	SXYPA	7.60	-1.50	4,478.00	7.20	7.80	9,624.00
SXYDB	12.80	+2.70	1,895.00	12.20	12.90	24,616.00	1,310.00	SXYPB	9.50	-2.00	982.00	9.10	9.50	6,692.00
							1,311.56							
SXYDC	10.00	+2.50	3,432.00	9.20	9.90	10,329.00	1,315.00	SXYPC	11.80	-2.70	104.00	11.40	12.40	898.00
SXYDD	7.40	+1.70	3,566.00	7.10	7.50	11,403.00	1,320.00	SXYPD	14.10	-2.70	744.00	14.00	14.50	4,367.00
SXYDE	5.50	+1.30	14,490.00	4.90	5.50	50,337.00	1,325.00	SXYPE	16.80	-3.80	528.00	16.90	18.10	3,331.00
SXYDF	4.20	+1.30	1,092.00	3.50	4.20	11,412.00	1,330.00	SXYPF	23.50	-0.20	11.00	20.30	21.80	796.00

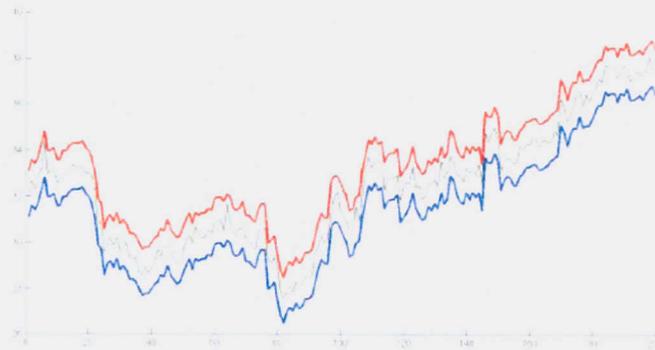
69days(2000-01--2003-11): We test 6,200,828 cases

17 cases:  
(A5) Violated

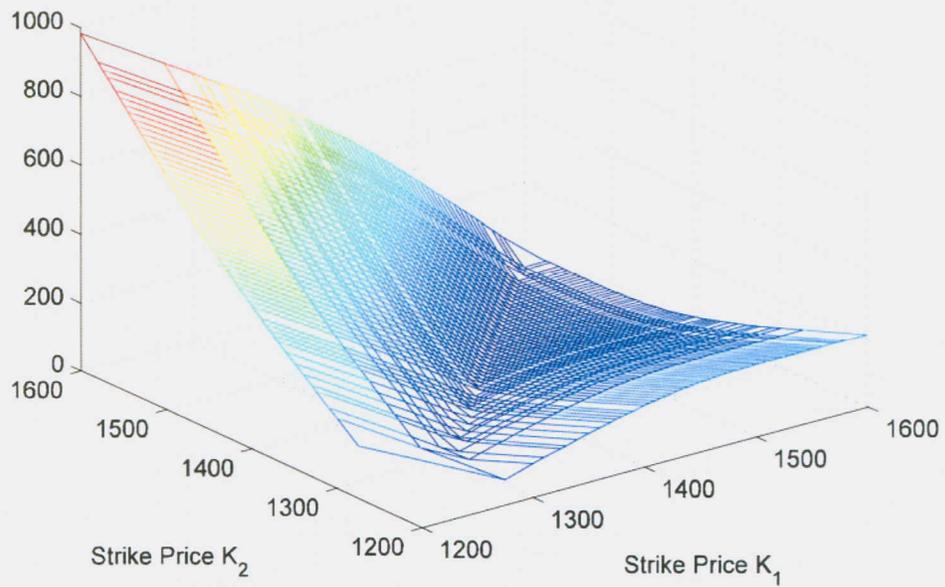
X-Y	$E^m[X] - E^m[Y]$	$E^{\mu}[X - Y]$	T-t (year)
Put-Put	4.5	-0.05	0.01917
Put-Put	19.2	-0.03	0.01917
Put-Put	54	40.4	0.01917
Put-Put	1.1	-0.00005	0.03835
Put-Put	0.1	-0.005	0.04657
Put-Put	0.1	-0.02	0.04657
Put-Put	0.05	-0.001	0.04657
Put-Put	0.05	-0.01	0.04657
Call-Call	141.2	133.8	0.00273
Call-Call	216.2	214.1	0.00273
Call-Call	241.2	240.9	0.00273
Call-Call	126.2	117.8	0.00273
Call-Call	201.2	198.1	0.00273
Call-Call	226.2	224.8	0.00273
Call-Call	236.2	235.5	0.00273
Put-Call	0	-0.02	0.00273
Put-Call	0	-0.01	0.00273

■ 一般的期权市场存在三种价格

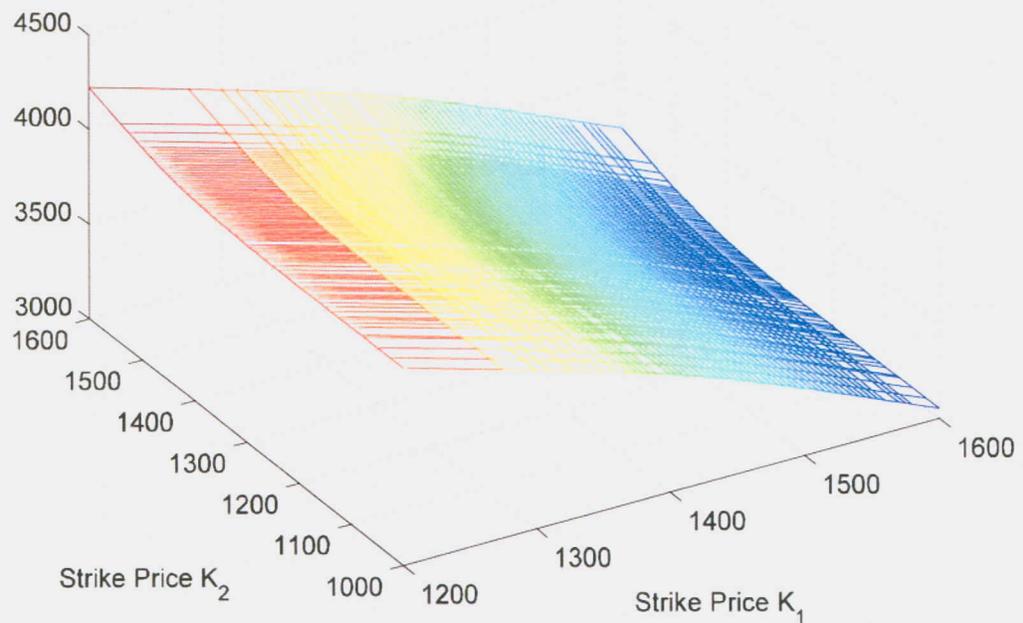
- 卖价 (Ask Prices)
- 买价 (Bid Prices)
- 成交价 (Last)



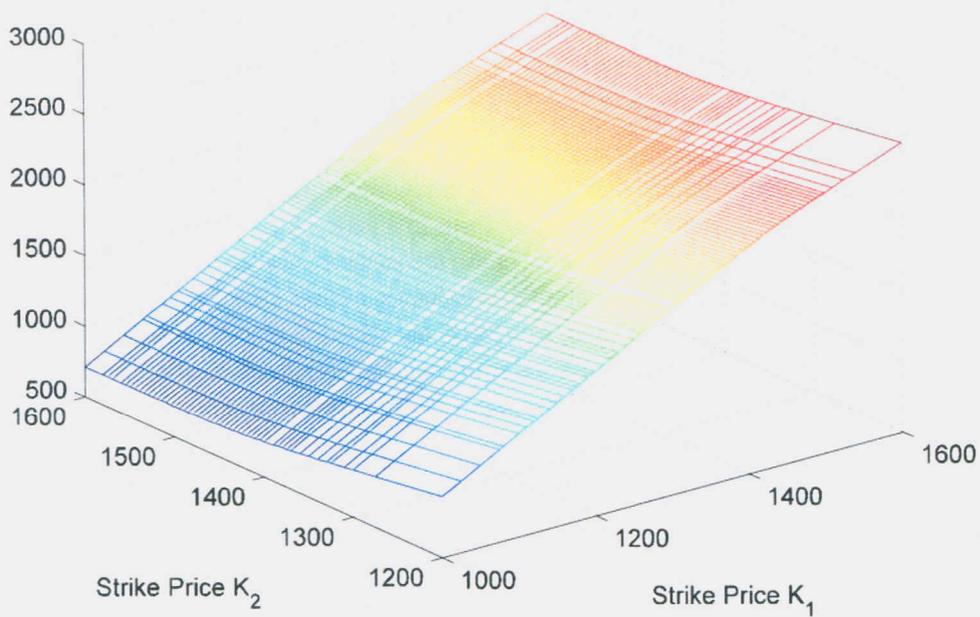
$$E^Q_\mu[(X_T - K_1)^+ - (X_T - K_2)^+] - (E^m[(X_T - K_1)^+] - E^m[(X_T - K_2)^+])$$



$$E^Q_\mu[(X_T - K_1)^+ - (K_2 - X_T)^+] - (E^m[(X_T - K_1)^+] - E^m[(K_2 - X_T)^+])$$



$$E^g_{\mu}[(K_1 - X_T)^+ - (X_T - K_2)^+] - (E^m[(K_1 - X_T)^+] - E^m[(X_T - K_2)^+])$$



$$E^g_{\mu}[(K_1 - X_T)^+ - (K_2 - X_T)^+] - (E^m[(K_1 - X_T)^+] - E^m[(K_2 - X_T)^+])$$

