# MODELLING THE EFFECTS OF COUPLER SLACK IN LONG COAL AND ORE TRAINS

## 1. The problem

Conventional couplings between trucks in coal and ore trains have a small amount of slack leading to some free relative motion of trucks in the longitudinal direction. This slack is typically around 25 mm, varying up to about 40 mm in worn couplers. The total slack in a train of say 100 trucks may thus be about 3 metres, which may lead to speed differentials along a train and to high stresses in trucks and couplers with consequent damage. The stresses can be severe enough for trains to split in two through destruction of a coupler or occasionally the breaking of a truck into two portions. This train splitting occurs on average about once per week Australia wide.

Tests from the USA on a new style *slackless* coupler indicate a major reduction in such longitudinal stresses. A reduction of 90% has been quoted, but the precise circumstances leading to this figure are not clear. Railways of Australia (ROA) is planning to test such couplers in the near future. In the meantime, the Mathematics-in-Industry Study Group (MISG) was asked if it could provide some mathematical insight into the problem. Four railway engineers and about ten mathematicians took an active interest in the problem at the MISG.

## 2. The direct approach

It is straightforward to write down Newton's equations of motion for a train incorporating general force functions for couplers, air resistance, rolling resistance, engine and braking forces, curving resistances and effects of terrain. Given realistic functions for such forces, one could perform long run simulations of typical train journeys by numerical solution of the equations of motion, and process the results in a suitable way. Different combinations of couplers could then be compared.

The opinion at the MISG was that such a program was quite feasible in principle over a longer term, but it met with certain immediate difficulties:

1. Force functions for slackless couplers were not yet available. A very thorough study of force *versus* extension and compression curves and their dependence on rates of change, was seen to be vital.

2. There seemed to be limited data on forces in couplers experienced in the field during typical journeys. Such data was seen as important for the validation of the mathematical model.

3. Since the train splitting phenomenon is a rare event per kilometers of travel, it might be difficult and time consuming in a simulation to identify, purely by chance occurrence, those unusual combinations of circumstances producing the phenomenon.

Because of the third point, an attempt was made at the MISG to identify and analyse those unusual circumstances, using simple models.

## 3. Linear spring models

A number of simple models were postulated and analysed at the meeting. A model where slack couplers are represented by damped, linear (Hooke's law) springs and slackless couplers by rigid connections between trucks yields linear differential equations. The model analysis is covered by textbook theory common to many oscillating systems. The general conclusion reached at the MISG was that changing the mixture of slack and slackless couplers had little effect on coupler stresses. The lack of realism in the model, however, inspired little confidence in the usefulness of this conclusion. Above all, it was thought that slackness demanded a non-linear treatment.

## 4. Impact models

An extreme example of a non-linear model is illustrated in Figure 1. Trucks are represented by discs and a slack coupler by an inextensible string or chain of length $s$. Figure 1a shows an *extensive impact* between two trucks. Initially the coupler is in an intermediate state and the trucks have a nett velocity of separation. When the inter-truck distance reaches value $s$, there is an impulse applied to both trucks (an infinite force in a zero time interval). If the chain is elastic (with infinite stiffness), the trucks rebound while conserving energy and momentum; thus they exchange velocities. If the chain is perfectly inelastic, energy is lost and the chain remains fully extended until a neighbouring impact changes its state. Intermediate cases of partial elasticity can be modelled by Newton's law of impact incorporating his *coefficient of restitution*. *Compressive impacts* illustrated in Figure 1b follow analogous rules. In the following four sections, we investigate a number of simple impact models.

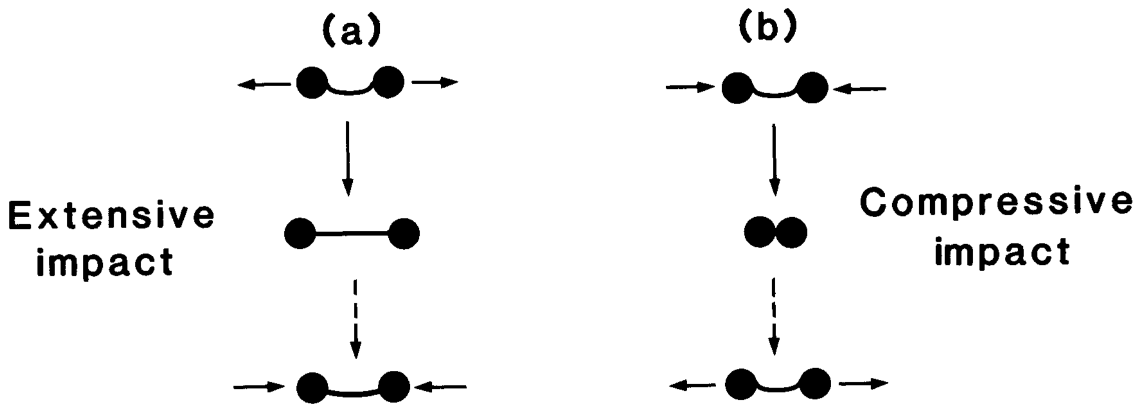


Figure 1: Compressive (a) and extensive (b) impacts in simple-impact models for conventional couplers.

## 5. Simulations with elastic impact models

Figure 2 illustrates the behaviour of a train with 10 units (which may each consist of several slacklessly coupled trucks) plus the engine. Initially the couplers are taken as fully compressed. The length of the trucks is taken out of the Figure, so that initially they are represented by a point at the origin $x = 0$. The engine then pulls away with constant acceleration of 1 and the couplers have slack $s = 1$. The figure shows the sequence of resulting impacts where units are progressively set in motion. Notable features are the chaotic *waves* or *pulses* of extensive impacts which travel to the back of the train, are reflected and travel forward as chaotic compressive waves. Notably too, the back unit begins to move when the length of the train is about 7.5, *not* its full length of 10. Obviously these effects are damped if one puts some degree of inelasticity in the model couplers, and we shall consider the extreme case of complete inelasticity in Section 6.

We summarise some results from 3 such simulations with different amounts of slack. In each case, the total train mass was 10 and the simulation was run for time 50. If $I_n^e$ and $I_n^c$ are the total extensive and compressive impulses received by coupler $n$ (counting from the front) during this time, then Table 1 lists the means

$$I^e = \frac{1}{N}\sum_1^N I_n^e$$

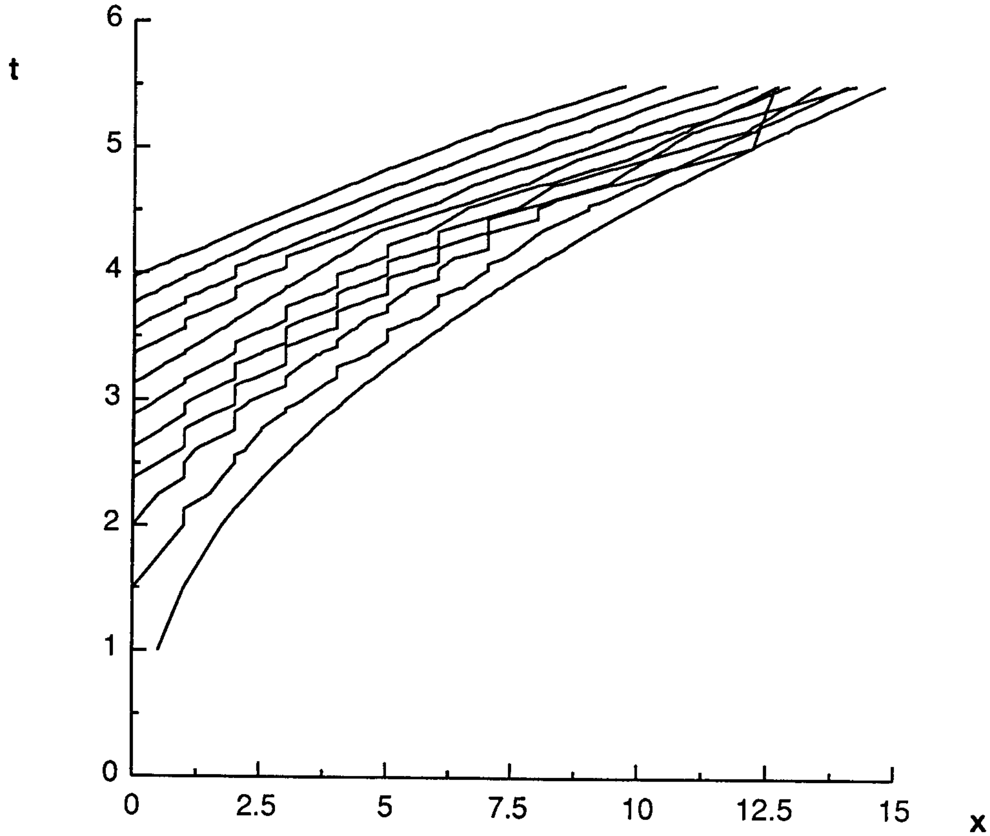


Figure 2: Simulation of a 10 unit train with perfectly elastic, simple-impact couplers, showing positions $x$ of units at time $t$.

and

$$I^c = \frac{1}{N}\sum_1^N I_n^c,$$

where $N$ is the number of slack couplers. As well as $N = 10$ we took $N = 20$ and $N = 50$. Table 1 gives some results.

Table 1

| simulation | # slack couplers | $I^e$ | $I^c$ |
|---|---|---|---|
| 1 | 10 | 2841 | 1399 |
| 2 | 20 | 1780 | 889 |
| 3 | 50 | 899 | 737 |

This shows that impulses on average increase as slackness is taken out of the train. More detailed examination of the simulation results confirms this trend for individual couplers.

The result may seem contrary to the USA results. However, in our simulations, no large impulses were experienced by couplers, so the fact that these impulses increase when slack is taken out may not be of major importance. We shall find below (Sections 7 and 8) that the large impulses occasionally generated in some scenarios are greatly reduced by taking out slack. Before demonstrating this, we first analyse our present simulation scenario for the case of perfectly inelastic couplers, to see if our conclusion is at all robust.

## 6. Steady pulling of a fully compressed train with perfectly inelastic couplers

We consider $N$ units of equal mass $m$, where each unit may be a single truck or a number of trucks which are rigidly connected by slackless couplers. Thus the system has $N$-1 conventional slack couplers, each with a slack (or free play) $s$. The front mass, which may comprise engines plus some trucks, pulls away with constant force $F$. Again we take out the length of trucks, so all are initially coincident at the origin. Because of the inelasticity, a typical configuration after a short time is like that shown in Figure 3. A number, $n$ say, of units at the front have the same speed while the remaining $N-n$ are at rest. The slack between the first $n$ units is fully taken up. The slack between units $n$ and $n+1$ is partly taken up. Henceforth, when we mention the $n$th coupler, we mean the $n$th *conventional* coupler, counting always from the front of the train.


Figure 3: Train and coupler states resulting from steady pulling (with force $F$) of a train that is initially fully compressed, and has perfectly inelastic, simple-impact couplers.

For fixed $n$, the centre-of-mass $x$ of the front $n$ units satisfies

$$\ddot{x} = F/nm, \tag{1}$$

so that, at time $t$,

$$x = \frac{1}{2}\, t^2 F/(nm). \tag{2}$$

Taking $t_1$ as the time when the $n$th coupler in Figure 3 becomes fully extended, we see that then

$$x = \frac{1}{2}(n+1)s, \tag{3}$$

so that

$$t_1 = [n(n+1)ms/F]^{1/2}. \tag{4}$$

Then the centre-of-mass has speed

$$\dot{x} = t_1 F/(nm) = \left[\frac{(n+1)Fs}{nm}\right]^{1/2}. \tag{5}$$

At time $t_1$, the front $n$ units act effectively as one mass $nm$ which makes an extensive impact with $(n+1)$th mass. Conservation of momentum and the inelasticity of the couplings then implies that, immediately after this, the front $(n+1)$ units move in unison with speed $V$, say, given by

$$nm\dot{x} = (n+1)mV. \tag{6}$$

The $n$th and $(n+1)$th units receive equal and opposite impulses

$$\begin{aligned} I &= nm(\dot{x} - V) = mV \\ &= [n/(n+1)]m\dot{x} \\ &= \left[\frac{n}{n+1}Fsm\right]^{1/2}, \end{aligned}$$

so that

$$I \simeq (Fsm)^{1/2} \quad \text{for} \quad n \gg 1, \tag{7}$$

where we used (5) and (6). This impulse is felt in the $n$th coupler.

Now suppose we vary the total amount of slack in the train by varying $m$ and $N$ while keeping the total mass

$$M = Nm \tag{8}$$

of the train constant. Then (7) gives

$$I \simeq (Fsm)^{1/2} N^{-1/2}, \tag{9}$$

which increases as the amount of slack $(N-1)s$ (or effectively $N$) decreases. This result is in qualitative agreement with the simulation result of the preceding section, in spite of the very different coupler characteristics. One therefore has rather more confidence in the prediction. The practical implication is that one might find more wear, fatigue and general progressive damage by attrition in the remaining (smaller number) of slack couplers, when some slackless couplers are introduced.

We note some technical points of interest in the preceding analysis. For large enough $n$, the velocity (5) of the front portion of the train is independent of $n$. Small accelerations occur between extensive impacts but these are almost exactly compensated for by the impacts themselves. Increasing $F$ simply increases this essentially constant velocity. Equation (7) shows that each coupler experiences essentially the same impact.

## 7. A train splitting scenario under engine power with inelastic couplers

Suppose a train in motion is in the state illustrated in Figure 4a, where the front $n$ couplers are fully compressed and the remainder fully extended. Assuming inelastic couplers, the dynamics is the same as in Section 6 until the state in Figure 4b is reached. Then one effectively has a mass $(N-n)$ moving at the original speed $v$ and a mass $nm$ moving at speed $v$ + (5). Evidently a large extensive impulse will be felt in coupler $n$ at $P$. By conservation of momentum, we find the impulse to be

$$I = \frac{1}{N}(N-n)n\left[\frac{n+1}{n}\,Fsm\right]^{1/2}$$

or

$$I \simeq \frac{1}{N}(N-n)n(Fsm)^{1/2} \quad \text{for} \quad n \gg 1. \tag{10}$$

This impulse is largest if $n = N/2$ in the initial state Figure 4a. This gives an impulse

$$I \simeq \frac{1}{4}(Fsm)^{1/2}N. \tag{11}$$

Again let us vary the amount of slack $N$ according to (8), giving

$$I \simeq \frac{1}{4}(FsM)^{1/2}N^{1/2}. \tag{12}$$

In sharp contrast with the result (9), this impulse *decreases* as the total slack $N$ decreases, showing *an advantage in removing slack from trains.* We note too that (12) is $N/4$ times that in (9). Since the much larger impulses (12) are potentially much more damaging than those in (9), it is relatively much more important that a reduction in impulses is now obtained by reducing train slackness.

ROA has suggested reducing $N$ from 100 to 20. This represents a 55% reduction in $I$. Smaller values of $N$ make it difficult to accelerate trains from rest and to reconfigure trains by removing or replacing portions of the train.

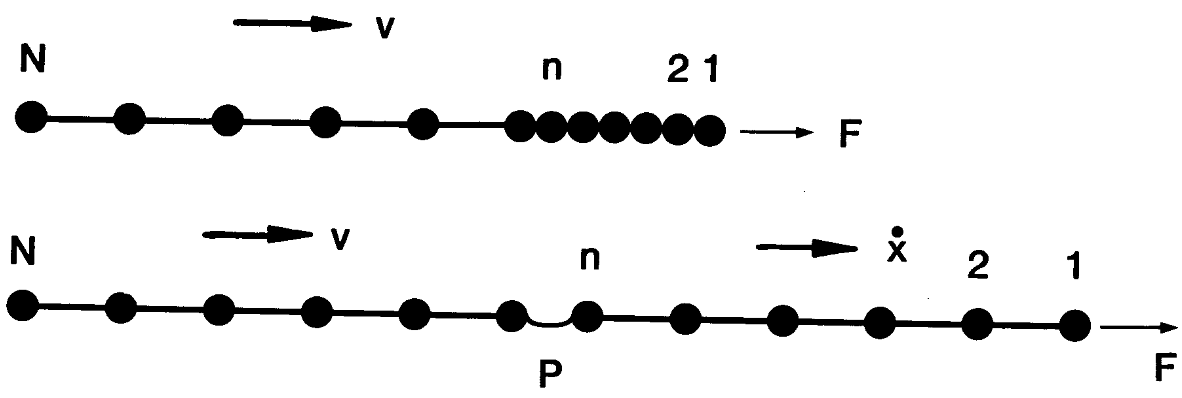


Figure 4: Initial (a) and later (b) configurations of a train with perfectly inelastic couplers in the scenarios studied in Sections 7 and 8.

The impact model, strictly speaking, implies instantaneous impulses and hence infinite forces of zero duration (formally the forces are multiples of Dirac delta functions). Let us relax this assumption and imagine that the force acts over the time

$$\Delta t \simeq (sm/F)^{1/2} \tag{13}$$

taken to extend the middle coupler. Then a rough estimate of the force in the coupler is

$$F_c \simeq I/\Delta t \simeq \frac{1}{4}FN. \tag{14}$$

Note first that this shows an even greater advantage ($N$ rather than $N^{1/2}$) in reducing slackness: an 80% force reduction for ROA's suggested reduction in slackness. Second, note that two engines each pulling with a 50 tonne force gives $F$ = 100 tonnes. For a completely slack train with $N = 100$ we then have $F_c \simeq 2,500$ tonnes showing this to be a likely train-splitting scenario. Further, if $s = 25$ mm and $m = 100$ tonnes, the two train portions in Figure 4b have a relative velocity

$$(Fs/m)^{1/2} \simeq 0.5 \text{ metre/sec}^2,$$

which is quite large for two masses each of 5000 tonnes.

Figure 5 illustrates how a scenario like Figure 4a might arise. Gravity tends to move trucks in the directions indicated by the arrows, with resultant compressions and extensions.

The prediction, that $n = N/2$ gives the largest impulse, implies that trains should split near the middle in this scenario. We did not have at our disposal data on train splittings, but ROA reported that railway *folklore* holds that $n \approx (2/3)N$. We obtain a value nearer to this in the next scenario.

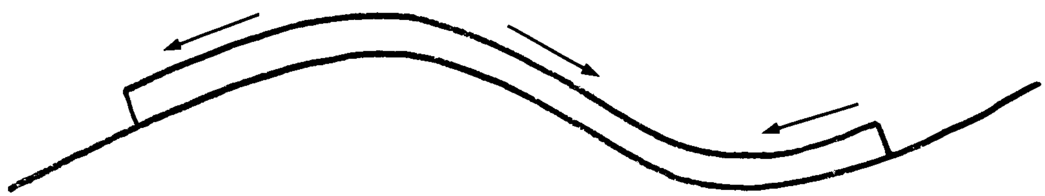

Figure 5: A terrain condition which might lead to the initial train configuration shown in Figure 4a.

## 8. A train splitting scenario for releasing of brakes with inelastic couplers

A substantial proportion of train splittings occur when brakes are released. This is believed to be due to a design limitation of braking systems, which results in a slow release of brakes progressing from the front to the back of the train, with a delay of about 1 second between trucks. Consequently, the front of the train tends to run away from the back, creating tension and impacts in couplers. Normally this is not serious because the impacts are small and shared between many couplers, somewhat in the manner of Section 6.

Suppose, however, that the train is in the condition of Figure 4a when brakes are first released. Assume the same inelastic model and represent the brake force on the front $n$ units at time $t$ by

$$F = -nm\mu + \lambda t, \tag{15}$$

where $\lambda$ and $\mu$ are independent of $m$ and $n$. One way to obtain such a force is as follows. Suppose that brakes on individual trucks are either fully on or fully off. Suppose there is a delay $\tau$ between release on neighbouring trucks. Then there is a delay $(m/m_0)\tau$ between successive units, where $m_0$ is the (fixed) mass of a truck. Thus at time $t$, the number of released brakes is the largest integer $k$ smaller than

$$m_0 t/(m\tau). \tag{16}$$

If each truck exerts braking force $\phi$, then a unit exerts braking force $(m/m_0)\phi$. Hence

$$F = -(n-k)(m/m_0)\phi,$$

which reduces to (15) on putting $\mu = \phi/m_0$, $\lambda = \phi/\tau$ and approximating $k$ by (16) itself.

We have given this rather ponderous derivation of (15) mainly to clarify the correct $m$ and $n$ dependence, because this aspect is crucial when train slackness is changed. Evidently (15) can be derived from more gradual release rules for individual brakes, but we omit the details.

After a suitable time, the train will reach the condition shown in Figure 4b. Suppose that all brakes are still on in the back $N-n$ units, while some of the rearmost brakes in the front $n$ units may still be on. (For (15) to be consistent with this assumption requires that some constraint be placed on $\lambda$ and $\mu$: see (18) below.) Then the centre-of-mass $x$ of the front portion has equation of motion

$$nm\ddot{x} = -nm\mu + \lambda t,$$

giving

$$x = vt - \frac{1}{2}\mu t^2 + \lambda t^3/(6mn).$$

The $(n+1)$th unit has position $y$ satisfying

$$(N-n)m\ddot{y} = (N-n)m\mu$$

and giving

$$y = vt - \frac{1}{2}\mu t^2.$$

At the time, $t_1$ say, when coupler $n$ at $P$ in Figure 4b becomes fully extended,

$$x = y + \frac{1}{2}(n+1)s,$$

which gives

$$t_1 = [3n(n+1)ms/\lambda]^{1/3}.$$

Thus the $n$th coupler feels, at $t_1$, an impulse

$$I = N^{-1}(N-n)nm[\dot{x}(t_1) - \dot{y}(t_1)]$$

or

$$I = \frac{1}{2}N^{-1}(N-n)n^{4/3}\lambda^{1/3}(3ms)^{2/3} \quad \text{for} \quad n \gg 1.$$

This is the largest when

$$n = (4N/7),$$

a result fairly close to the *folklore* value $(2N/3)$.

Now let us vary train slack according to (8). This gives maximum impulse

$$I = AN^{2/3}, \tag{17}$$

where $A$ is constant. This impulse is reduced when slack is removed from the train: the improvement is better than in the scenario of Section 7 ($N^{2/3}$ instead of $N^{1/2}$). For ROA's proposal of reducing $N$ from 100 to 20, $I$ is reduced by 66%.

By our previous method of estimating the force $F_c$ in coupler $n$, we get

$$\begin{aligned} F_c &\simeq I/\Delta t \\ &= I[\dot{x}(t_1) - \dot{y}(t_1)]/s \end{aligned}$$

or

$$F_c \simeq BN^{4/3},$$

where $B$ is constant. This shows a marked reduction when slack is removed from the train. For ROA's proposed slack reduction, $F_c$ is reduced by 88%, close to the value of 90% quoted in the USA report.

Briefly, the technical condition for (15) to be consistent with the braking assumptions is that

$$F(t_1) \equiv -nm\mu + \lambda t_1 \leq 0,$$

which requires

$$\lambda < \frac{1}{3}(4\mu/7)^3\frac{M^2}{Ns} \tag{18}$$

for all $N$ considered. A slightly modified analysis is required when this does not hold.

If the brake release occurs at changing time intervals between trucks, we might obtain in place of (15) a brake force

$$F = -nm\mu + \lambda t^{\alpha}.$$

Then $\alpha > 1$ implies that the time delay between trucks $k$ and $k+1$ decreases with $k$ like $k^{1-\alpha}$. Carrying through the above analysis then gives a maximum impulse in the coupler number

$$n = \frac{2}{3}(1 - \frac{1}{3\alpha+4})N,$$

which is close to the folklore value $(2N/3)$ for $\alpha$ much bigger than 1.

## 9. Conclusions

Our results constitute only a first step in the analysis of a complex problem. The assumptions and simplifications used mean that the results are somewhat removed from the real situation, and so can be regarded only as a basis for discussion and further work. With this proviso, the main conclusions are summarised by Table 1 and equations (9), (12) and (17). They indicate that small impulses in couplers tend to be increased by removing slack from trains (by incorporating slackless couplers), while rare but exceptionally large impulses (producing severe damage or even train splitting) tend to be substantially reduced by taking slack out of trains. The latter conclusion seems consistent with field measurements from USA.

The conclusions reached here could be tested by a suitable research program. Initially, a further phase of exploratory modelling would be valuable. This would still involve many simplifying assumptions, but would examine a wider range of rather more realistic scenarios, using both simulations and mathematical analysis, and make some comparisons with data. Records of measured large stresses, of damage to couplers and trucks and of train splitting events and the circumstances in which they occur would be valuable here.

In a later, more protracted, phase of research, forces on trucks and couplers would be modelled in a more realistic way, as outlined in Section 2. This would require data listed in items 1 and 2 of Section 2. Computer simulations would then focus on those events that the initial phase of research revealed to be important.

## Reference

H. Goldstein, *Classical mechanics* (Addison-Wesley, Mass. 1973).