Models of Consumer Behaviour

Problem presented by

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Problem statement

The problem posed to the Study Group was to construct models for consumer behaviour that might be useful in tools for brand management in markets for fast-moving consumer goods. Such models must take into account various psychological and sociological factors that describe respectively how consumers are influenced by what is on sale and who else is already buying. The outputs of the models should be predictions for the division of market share between competing products. Two phenomena of particular interest for assessing modelling options are the 'decoy effect' and 'lock-in'.

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Summary

The Study Group looked at a variety of modelling frameworks for investigating market behaviour in fast-moving consumer goods. Section 1 presents the background and objectives to the work and the remaining sections of the report develop particular types of model, as follows.

- **Sections 2 and 3** consider consumer loyalty and psychology within lumped, deterministic models. Key psychological features such as 'minimising anticipated regret' are included in the models. It is seen how the introduction of a new product can explain behaviour such as the 'decoy effect'.
- **Section 4** adds sociological aspects, for example the positive influence of friends already buying particular products, and shows how they can lead to the phenomenon of 'lock-in' if the social interactions are sufficiently strong.
- **Section 5** adopts a probabilistic approach, based on Markov chains, to the study of both psychology and sociology. The qualitative conclusions of the earlier deterministic models are confirmed, namely that new products can give rise to a decoy effect, and social interactions can lead to lock-in.
- **Section 6** presents a different framework for studying social interactions, in which consumers are explicitly given individual psychological characteristics and so behave in intrinsically different ways. In this model, the probabilities of individuals buying particular products change over time, under the influence of other members of the population. Simulations, supported by analysis, again exhibit lock-in for sufficiently strong interactions.
- **Section 7** looks in detail at the decision process of individual consumers, as a sequence of pairwise comparisons of different products. It is found that a two-level decision tree can lead to a decoy effect. When loyalty is strong, the decoy effect persists, even though the decoy product gains almost zero market share. In contrast, the standard Logit model for consumer choice does not exhibit the decoy effect. Preliminary network simulations using a two-level decision process exhibit lock-in, which can be prevented by advertising.
- **Section 8** also looks at network effects, but this time at the propagation of consumer behaviour and the way in which new products gain market share. Simulations show how the final market shares depend heavily on the position of the new product relative to the existing ones in quality space.
- **Section 9** gathers together some further ideas for modelling frameworks. In a particle-dynamics model, consumers could be modelled as particles moving in product space (corresponding to changes in their preferences), under the influence of 'potentials' that are due to the available products. Again, a decoy effect seems possible. Another possibility is to model a continuum of products, so that market shares are replaced by a density in product space, which evolves over time.

1 Background and introduction

1.1 General intentions

Consumer products such as shampoo or tomato sauce are designed so that they appeal to consumers, encouraging them to buy those products. To that end, the industrial R&D organisation tends to focus on understanding and manipulating product attributes. However, buying behaviour is not only a function of the product: it is also, and in some cases perhaps more so, a function of the consumer, his social environment of other consumers, the competing products in the marketplace, and the brand marketing strategy. In order to design the best product, it is necessary to understand not just the physics and chemistry of the product, but also the psychology of consumers and the sociology of consumer groups or networks.

The goal of this study is to have a model of the marketplace that describes certain aspects of consumer buying behaviour. There are two main parts to such a model:

- A description of a population of 'consumers', who each choose (buy) repeatedly one of a number of competing 'brands' (we can ignore the difference between product and brand for these purposes). This subdivides into a description of the behaviour of a single consumer ('psychology'), and of the collective behaviour of a group, in other words of the interactions between consumers ('sociology').
- A description of 'brand management', *i.e.* the strategy of brand managers when changing the attributes of a brand such as price or quality in response to events in the marketplace.

Traditional marketing models tend to focus on the second element, and treat the large number of consumers in a macroscopic, averaged way. Alternatively, one can look at individual consumers and their buying behaviour, and try to derive observable large scale effects, like changes in market share. Ideally one would like to connect the 'microscopic' consumer viewpoint to the 'macroscopic' viewpoint of the brand manager.

1.2 Factors in the models

The main features which were included in the various models are:

$\mathcal{A}$. Loyalty

Loyalty is the tendency for (some) consumers to stick to the same products. With this as a key effect, deterministic, continuous-time models will be systems of ordinary differential equations; the stronger the loyalty, the slower the changes in numbers of people buying particular products. For discrete-time models, the degree of loyalty corresponds to the size of diagonal elements in a transition matrix. On the other hand, with no loyalty (or influence of other people) whatsoever, market share — or chance of someone making a particular purchase — has no dynamic behaviour and would instead depend only upon what is currently on the supermarket shelves.

Another aspect of loyalty, not allowed for in our models so far, would be a memory effect, to represent people returning to products they had previously used, after trying something new they then didn't like. This could be taken into account perhaps by using recurrence relations or differential equations of higher than first order (or even employing delay-differential equations).

$\mathcal{B}$. Sociology

Sociology in this context is concerned with how one person's buying is influenced by that of others. With some sort of tendency of people to buy the same brands, there is a possibility of 'lock-in', with one product dominating the market, even if its competitors have more or less identical 'qualities' (including price). This effect and its opposite, people wanting to be different, are easily modelled by ODE and discrete-time models.

$\mathcal{C}$. Psychology

Psychology covers what, and how, aspects of the actual items on the shelves influence people to make their choices, possibly buying something different from previously. (Advertising might be subsumed into these characteristics but could also possibly be considered as part of the sociological influences, especially if the advertising takes the form of a well known figure endorsing a product.) More specifically, the following four properties have been identified by Unilever as being important and their influences were included in one or more models:

1. *Minimise anticipated regret.* This refers to how just two products compare with each other as regards different qualities, which can include price (or 'affordability' = 1/price). A consumer might judge one item to be superior to another in all respects. The first is then a safe choice for the consumer.

2. *Attribute change.* The introduction of a new product onto the market can change the way consumers, or at least some of them, view established brands. This might be by drawing attention to some quality which was not previously much regarded, or it might make people give different weightings to the (established) qualities when making their decisions. The former can be considered to be a special case of the latter.

3. *Outlier avoidance.* When a number of products are in many aspects quite similar, there can be a tendency for people to avoid 'strange' ones, *i.e.* others which are substantially different from the majority in price or some other respect. Items near the average can be favoured.

4. *Decision process change.* A straight choice between two items might be relatively easy; they can be compared according to price, size *etc.* and a decision made. With three or more, comparisons might be made between two things at a time, one could be eliminated and then the winner contrasted with a third.

1.3 Possible predicted effects

The models which were looked at during the course of the Study Group were intended to shed light on:

1. **How a 'decoy' product might influence the market.** The appearance of a third product might significantly change the market shares of two others, while getting minimal sales itself. This effect is one of the most robust biases in consumer choice, and has been observed in product classes from chocolate bars to TV sets to beer. The decoy effect illustrates the importance of consumer psychology, of understanding how consumers perceive products, and how consumers judge quality prior to purchasing the product.

2. **The dynamics of market share;** how sales of products can vary over time. For example, even if two products are equal in all relevant aspects, then after a long time of consumer activity it might be that each product takes 50% market share (preserving the symmetry), or one product takes nearly 100% market share (breaking the symmetry), or that there is no steady state, with market dominance alternating between the two brands. The second of these three cases is called 'lock-in', corresponding to one brand obtaining a virtual monopoly, which is almost impossible to break.[1]

3. **How a new product will fare,** given its quality profile compared with existing brands. This question is complementary to that of the decoy, asking what market share a new product will gain rather than how it will affect the market shares of existing products.

4. **'Choice overload':** when there are just too many possible options for potential consumers to pick from, and many will walk out of the shop without making a purchase. This possibility was considered only briefly during the Study Group, since the focus was on market share and it was assumed that the types of product being modelled were 'consumer staples', so that every consumer would make some purchase.[2]

1.4 The models

The main thrusts of the week's activities were in building a handful of quite specific models, into which the above psychological and sociological influences could be fed, and

[1] Unilever have previously carried out simulations on a probabilistic, discrete-valued, discrete-time, agent-based model, with a finite number of products (two) and one type of consumer (*i.e.* all the agents behaving the same way). This model is a version of the famous Ising model in statistical mechanics. Lock-in here corresponds to a phase transition in statistical mechanics.

[2] There was a brief discussion of the possibility of taking the number of products to influence people's choices directly. It was eventually decided, however, that this did not really need to be included as for every extra product on the market there would be another loss term for the sales of each existing product.

in seeing how these aspects $\mathcal{B}$ and $\mathcal{C}.1$ – $\mathcal{C}.4$ should be represented. Various types of overall model were looked at. The sorts which might be considered could be mainly (but probably not exclusively) categorised according to whether they were:

deterministic (*e.g.* how a market share evolves for high overall sales)	or	probabilistic (a small market should be treated as a random process)
continuous valued (*e.g.* market share)	or	discrete valued (*e.g.* what one consumer buys)
continuous time	or	discrete time
lumped model (populations buying different brands)	or	agent model (individuals doing the buying)
continuous product range	or	finite number of brands
identical consumers	or	consumers with different intrinsic behaviour (*e.g.* men or women)

Allowing for continuous or discrete agents, this already gives 96 types of model. Needless to say, only a few were looked at. As well as the basic types of model mentioned above, they can exclude or include sociological influences (making 192 types of model, or 288 if 'drivers', people of significantly more than average influence, are allowed for).

1.5 Preliminary modelling

The discussions in the Study Group started with the consideration of a simple probabilistic model. Given two products, the probability of a typical consumer buying the first at some point in time m is p_{1m} and of buying the second is $p_{2m} = 1 - p_{1m}$. The corresponding probabilities at the next time (possibly the following day or month) are

$$p_{1(m+1)} = \alpha^*_{11} p_{1m} + \alpha^*_{12} p_{2m} \quad \text{and} \quad p_{2(m+1)} = \alpha^*_{21} p_{1m} + \alpha^*_{22} p_{2m} \,. \tag{1}$$

Here (α^*_{ij}) forms a transition matrix ($\alpha^*_{11} + \alpha^*_{21} = 1 = \alpha^*_{12} + \alpha^*_{22}$) and the process is being represented as a Markov chain. The smallness of the off-diagonal elements α^*_{12} and α^*_{21} can be thought of as a measure of brand loyalty.

Going over to a continuous-time process (letting the time step shrink to zero with $\alpha^*_{11} \to 1$ and $\alpha^*_{22} \to 1$) the probabilities $p_1(t)$ and $p_2(t) = 1 - p_1(t)$ of buying each product now evolve according to a system of linear ODEs,

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = \alpha_{11} p_1 + \alpha_{12} p_2 \quad \text{and} \quad \frac{\mathrm{d}p_2}{\mathrm{d}t} = \alpha_{21} p_1 + \alpha_{22} p_2 \,, \tag{2}$$

where now $\alpha_{21} = -\alpha_{11}$ and $\alpha_{12} = -\alpha_{22}$.

Taking N people, all behaving identically but independently (no sociology for the moment), the expected market shares, $X_1 = p_1$ and $X_2 = p_2$, evolve in the same way, (2). Assuming that the numbers are large enough and that populations can be regarded as varying deterministically, the X's can be thought of as populations (or, rather, fractions of the population) and we have a lumped model for them:

$$\frac{\mathrm{d}X_1}{\mathrm{d}t} = \alpha_{11}X_1 + \alpha_{12}X_2 \quad \text{and} \quad \frac{\mathrm{d}X_2}{\mathrm{d}t} = \alpha_{21}X_1 + \alpha_{22}X_2\,. \tag{3}$$

These populations, or market shares, still satisfy $X_1 + X_2 = 1$. The off-diagonal coefficients α_{ij} determine the rate at which people change to product i from product j, $i \neq j$.

2 Psychology: influence of what is on the shelves

2.1 The effects to be included

All the aspects $\mathcal{C}.1$–$\mathcal{C}.4$ from Section 1.2 were looked at during the Study Group. The last of these (decision process change) is considered in detail in Section 7; this section concentrates on $\mathcal{C}.1$–$\mathcal{C}.3$.

1. Minimise anticipated regret. This modelling property was taken to lead to simple comparisons along the lines of 'with regard to quality k, is product i better than product j?' If the answer to all (relevant) questions, $k = 1, \ldots, n_q$, if n_q is the number of (pertinent) qualities, is no, then a consumer will not change from j to i. The more times the answer is yes, the faster such a change is likely to happen. This can be represented by taking the flow-rate constants to be of the form

$$\alpha_{ij} = N_{ij} = \text{number of qualities for which product } i \text{ is better than product } j.$$

This can alternatively be written as

$$\alpha_{ij} = \sum_k \mathrm{H}\left(Q_{ki} - Q_{kj}\right),$$

with H the usual Heaviside function ($\mathrm{H}(s) = 1$ for $s > 0$, $\mathrm{H}(s) = 0$ for $s \leq 0$) and Q_{ki} the measure of quality k for product i. Such laws can be, and in the following section are, generalised to make this effect more pronounced

2. Attribute change. This was eventually thought of in terms of a utility function that is dependent upon what products are actually on sale. With such a (positive) function $U(\boldsymbol{Q})$, capturing all relevant product qualities in a single value, product i is 'better' than product j if $U(\boldsymbol{Q}_i) > U(\boldsymbol{Q}_j)$. With this comparison, consumers are more likely to move from j to i than *vice versa*: $\alpha_{ij} > \alpha_{ji}$. This might suggest having

$$\alpha_{ij} = \frac{U(\boldsymbol{Q}_i)}{U(\boldsymbol{Q}_j)} \qquad \text{or} \qquad \alpha_{ij} = M_1 + U(\boldsymbol{Q}_i) - U(\boldsymbol{Q}_j)\,; \tag{4}$$

the positive constant M_1 is included in the second version to ensure that rates do not become negative.

If there are only two quality measures, Q_1 and Q_2, there is the useful concept of the 'trade-off line'. This might be fixed by having a known, linear utility function $U = \beta_1 Q_1 + \beta_2 Q_2$ for fixed (non-negative) β_k's. A trade-off line would then be a line of constant (positive) U. On the other hand, with just two products dominating the market, these two should have the same value of utility function, as they must, in some sense, be equal, and the trade-off line should pass between the positions representing these products in 'quality space'; see Fig. 1. The precise utility function, the coefficients β_k in this linear example, might themselves depend upon what is on the market – in particular the brands' positions

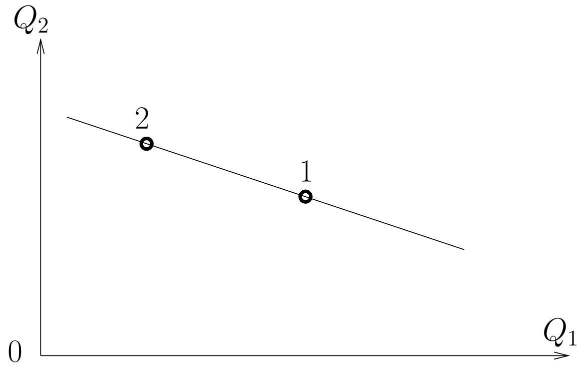


Figure 1: The trade-off line passing through the quality points for two existing products.

in quality space – as people's perceptions as to what is a good buy can be influenced by the attributes of the various products. The available products can define how people weight their qualities.

With at least as many products, n_p, as types of quality, n_q, the utility function could be given by

$$U(\boldsymbol{Q}) = \sum_{k=1}^{n_q} \beta_k Q_k \tag{5}$$

with the non-negative coefficients β_k chosen so that $U(\boldsymbol{Q}) = 1$ gives a 'best fit' to the products. For instance, one might take the β_k's to minimise

$$\sum_{i=1}^{n_p} \left(U(\boldsymbol{Q}_i) - 1\right)^2 ,$$

or else minimise a measure of distance such as

$$\sum_{i=1}^{n_p} \left(\sum_{k=1}^{n_q} \beta_k^* Q_{ki} - \beta \right)^2 \quad \text{or} \quad \sum_{i=1}^{n_p} \left| \sum_{k=1}^{n_q} \beta_k^* Q_{ki} - \beta \right|$$

for $\beta = 1/\sqrt{\sum_k \beta_k^2}$ and $\beta_k^* = \beta\beta_k$.

This sort of idea presumes that comsumers will do such sort of calculations (in their heads, or otherwise), or at least get a feel for what might be a reasonable utility function. It also leads to all consumers being much the same, with no allowance for the possibility of being price or quality-conscious. Such a characteristic might be catered for (to model different types of consumer) by allowing U to be nonlinear, for instance

$$U = \sum_k \beta_k Q_k^{\gamma_k} .$$

Another way of allowing for price consciousness might be to take

$$U(\boldsymbol{Q}) = U^*(\boldsymbol{Q}) + \beta_1^+ Q_1 = (\beta_1 + \beta_1^+)Q_1 + \sum_{k=2}^{n_q} \beta_k Q_k$$

with U^* taken to be the earlier U in equation (5), chosen to give the best fit for the existing products, and then adding a bias $\beta_1^+ Q_1$, with $\beta_1^+ \geq 0$, if Q_1 is 'affordability'.

3. Outlier avoidance. This effect means that consumers tend to change to products near a 'centre of mass'. In quality space the centre of mass would be at

$$\overline{\boldsymbol{Q}} = \frac{1}{n_p} \sum_{i=1}^{n_p} \boldsymbol{Q}_i \,,$$

the average of the qualities of the n_p products. The average might be weighted, perhaps according to how much shelf space the products take up, although this might instead be thought of as an aspect of 'sociology'. With a measure of 'distance' between products, for instance

$$d_{ij} = d(\boldsymbol{Q}_i, \boldsymbol{Q}_j) = \sum_{k=1}^{n_q} |Q_{ki} - Q_{kj}| \,,$$

products close to the centre of mass, *i.e.* those i with low values of $d(\boldsymbol{Q}_i, \overline{\boldsymbol{Q}})$ are to be preferred. [3]

A possible choice for the rate constants could now be

$$\alpha_{ij} = \frac{(M_2 + d(\boldsymbol{Q}_j, \overline{\boldsymbol{Q}}))}{(M_2 + d(\boldsymbol{Q}_i, \overline{\boldsymbol{Q}}))} \tag{6}$$

for some $M_2 \geq 0$. This constant can be included to stop α_{ij} being zero or infinity if $\boldsymbol{Q}_j = \overline{\boldsymbol{Q}}$ or $\boldsymbol{Q}_i = \overline{\boldsymbol{Q}}$, respectively.

2.2 Combining behaviour

The three effects discussed in Section 2.1 can be amalgamated to produce coefficients of the form

$$\alpha_{ij} = N_{ij}(M_1 + U(\boldsymbol{Q}_i) - U(\boldsymbol{Q}_j))(M_2 + d(\boldsymbol{Q}_j, \overline{\boldsymbol{Q}}))/(M_2 + d(\boldsymbol{Q}_i, \overline{\boldsymbol{Q}})) \,, \tag{7}$$

or

$$\alpha_{ij} = M_3 \exp(N_{ij} + U(\boldsymbol{Q}_i) - U(\boldsymbol{Q}_j) + d(\boldsymbol{Q}_j, \overline{\boldsymbol{Q}}) - d(\boldsymbol{Q}_i, \overline{\boldsymbol{Q}})) \,, \tag{8}$$

[3] The more standard Euclidean metric, $d(\boldsymbol{Q}_i, \boldsymbol{Q}_j) = \sum_{k=1}^{n_q} (Q_{ki} - Q_{kj})^2$, could of course be used, but the 1-norm version is taken here as more likely to be representative of the considerations of a casual buyer. In any case, this sort of distance has to be subjective in that different consumers will give different priorities to different aspects of quality.

to give but two possibilities. Other combinations, for instance with some rational terms as in (7) and some expontial terms as in (8), or sums rather than products, *e.g.*

$$a_1 N_{ij} + a_2(M_1 + U(\boldsymbol{Q}_i) - U(\boldsymbol{Q}_j)) + a_3(M_2 + d(\boldsymbol{Q}_j, \overline{\boldsymbol{Q}}))/(M_2 + d(\boldsymbol{Q}_i, \overline{\boldsymbol{Q}}))\,,$$

or more complicated functions, are clearly possible. Also, there is no need have symmetry built into these factors, for instance the attribute change (utility function) could appear through a factor or term $\exp(a_1 U(\boldsymbol{Q}_i) - a_2 U(\boldsymbol{Q}_j))$ and the outlier avoidance through $(M_2 + d(\boldsymbol{Q}_j, \overline{\boldsymbol{Q}}))/(M_3 + d(\boldsymbol{Q}_i, \overline{\boldsymbol{Q}}))$.

An explicit dependence upon the number of products on the market is easily included by introducing a factor $f(n_p)$. 'Product overload' can be built into the model by having $f(n_p) = o(1/n_p)$ for $n_p \to \infty$ so that the total rates of change decrease as the number of products goes up. An extra factor like $1/(M_4 + d(\boldsymbol{Q}_i, \boldsymbol{Q}_j))$ could also be included if people were thought not to tend to change to vastly different products.

2.3 Implications of the rate constants

The four effects to be investigated, as listed in Section 1.3 can now be looked at. Particular implications of the second and third facets of the psychology are discussed here. The first, minimizing anticipated regret, is considered in more detail in Section 3. The fourth, decision process change, is treated in Section 7.

Choice overload. Because rates of change go up with the number of terms, and hence products, choice overload is not predicted by any of the suggested forms of the coefficients α_{ij}, unless it is built in explicitly by including a factor $f(n_p)$ with f decreasing fast enough with the number of products, n_p.

Dynamics of market share. Market shares do not depend significantly, in their qualitative behaviour, upon the coefficients α_{ij}. However, the values of the α_{ij} will govern how fast the market shares tend towards their equilibrium values.

With constants such as in (7), $\alpha_{ij} \geq 0$ and $\alpha_{ij} + \alpha_{ji} > 0$ for $i \neq j$; without the factors N_{ij}, or using (8), $\alpha_{ij} > 0$ for $i \neq j$. The linear system,

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = \sum_j a_{ij} X_j\,, \tag{9}$$

where $a_{ij} = \alpha_{ij}$ for $i \neq j$ and $a_{ii} = -\sum_{j \neq i} \alpha_{ji}$, turns out to have a negative semi-definite coefficient matrix (a_{ij}), assuming only that $\alpha_{ij} \geq 0$ for $i \neq j$. This follows since, for any

$\boldsymbol{x}$ and, for simplicity taking $\alpha_{ii} = 0$ (no movement from i to i),

$$
\begin{aligned}
\sum_{i,j} x_i a_{ij} x_j &= \sum_{i,j} \alpha_{ij} x_i x_j - \sum_{i,j} \alpha_{ji} x_i^2 \\
&= \sum_{i,j} (\alpha_{ij} x_i x_j - \alpha_{ji} x_i^2) \\
&= -\tfrac{1}{2} \sum_{i,j} (\alpha_{ji} x_i^2 - (\alpha_{ji} + \alpha_{ij}) x_i x_j + \alpha_{ij} x_j^2) \\
&\le -\tfrac{1}{4} \sum_{i,j} ((\alpha_{ji} - \alpha_{ij}) x_i^2 + (\alpha_{ij} - \alpha_{ji}) x_i^2) = 0\,.
\end{aligned}
$$

Note that this inequality is strict if there are some i, j with $x_i \neq x_j$, unless $\alpha_{ij} = \alpha_{ji} = 0$. A change of co-ordinates to some $\boldsymbol{Y} = (Y_0, Y_1, \ldots, Y_{n_p-1})$ with $Y_0 = \sum X_i$ then gives $Y_0 = \text{constant} = 1$, for the total market share to be one, and a unique steady state for the revised, but equivalent, system

$$
\frac{\mathrm{d}Y_i}{\mathrm{d}t} = \sum_{j=0}^{n_p-1} b_{ij} Y_j \qquad (i = 1, \ldots, n_p - 1)\,,
$$

since for our coefficients we do know that $\alpha_{ij} + \alpha_{ji} > 0$ for $i \neq j$. It is then clear that (9) has a unique steady state $\boldsymbol{X}^*$ with $\sum X_i^* = 1$ and that this is a global attractor: for any initial data, $\boldsymbol{X}(t) \to \boldsymbol{X}^*$ as $t \to \infty$. That this is a 'sensible' solution, $X_i^* \geq 0$ for all i, is a consequence of $\frac{\mathrm{d}X_i}{\mathrm{d}t}$ being non-negative should $X_i = 0$ with $X_j \geq 0$ for other j: if ever the market shares X_i are all non-negative they must remain so and in particular $X_i^* = \lim_{t\to\infty} X_i(t) \geq 0$.

(For other sets of coefficients, not given by (7) or (8) for example, it is possible that $\alpha_{ij} = \alpha_{ji} = 0$ for some i and j with $i \neq j$. It is then conceivable that the shopping population can split into two (or more) camps, behaving quite independently so that the market shares associated with these camps remain constant. The steady states then form a one (or more) -parameter family and the long-term market shares will be fixed by the initial data. Such behaviour means that there would have to be two products such that, having bought one, someone would *never* buy the other.)

New products and decoys. The particular effects of 'attribute change' and 'outlier avoidance' are not especially surprising. Factors from the latter, something like

$$
(M_2 + d(\boldsymbol{Q}_j, \overline{\boldsymbol{Q}}))/(M_2 + d(\boldsymbol{Q}_i, \overline{\boldsymbol{Q}}))
$$

for α_{ij} indicate first of all that a totally new product, significantly different from existing brands so that its distance from the mean is large, will gain custom slowly and lose it rapidly. The introduction of a new product close to an existing one will move the centre of gravity towards the two, thereby pulling in consumers from elsewhere. This latter case would not count as a decoy, as the new product would acquire a similar market share to that of its neighbour. As an example, consider the case of initially just two products. With only this factor from outlier avoidance, say with $M_2 = 0$, $\alpha_{ij} = d(\boldsymbol{Q}_j, \overline{\boldsymbol{Q}})/d(\boldsymbol{Q}_i, \overline{\boldsymbol{Q}})$, and the steady market shares are $X_1 = X_2 = 1/2$. The introduction of a third product, with qualities identical to those of the first, leads to new equilibrium market shares $X_1 = X_3 = 4/9$, $X_2 = 1/9$. With such simple α_{ij}, a decoy could be applied to more dramatic effect by positioning it so that $\overline{\boldsymbol{Q}} = \frac{1}{3}\sum \boldsymbol{Q}_i = \boldsymbol{Q}_1$. Then X_1 becomes 1 immediately. Such behaviour can be ameliorated by reintroducing the constant M_2.[4]

The influence of attribute change, *e.g.* through $U(\boldsymbol{Q}_i)/U(\boldsymbol{Q}_j)$ for $U(\boldsymbol{Q}) = \sum \beta_k Q_k$ with the β_k's, along with a β, chosen to minimise $\sum_i(\sum_k \beta_k Q_{ki} - \beta)^2$ subject to $\sum_k \beta_k^2 = 1$, also leads to rather unsurprising behaviour. A new product coming onto the market will change the β_k's, unless it satisfies $U(\boldsymbol{Q}) = \beta$; it will do well if it is of good overall quality (a high value of U means rapid gain and slow loss of custom) and badly if it is of low overall quality (a low value of U means slow gain and rapid loss of custom). A more interesting point is a possible rôle in a decoy effect. For just two qualities and two products, $U(\boldsymbol{Q}) = \beta$ is a trade-off line running through the points corresponding to the products. According to our simple coefficients $\alpha_{ij} = U(\boldsymbol{Q}_i)/U(\boldsymbol{Q}_j)$, the two products have equal equilibrium market share. Introducing a third product a little 'below' the first changes the β_k's and β so that $U(\boldsymbol{Q}) \equiv \beta_1 Q_1 + \beta_2 Q_2 = \beta$ now passes between the first and third, but still through the second if the new product is positioned symmetrically (see Fig. 2). With the specific example of $\boldsymbol{Q}_1 = (5, 3)$, $\boldsymbol{Q}_2 = (2, 4)$, $\boldsymbol{Q}_3 = (3, 1)$, the

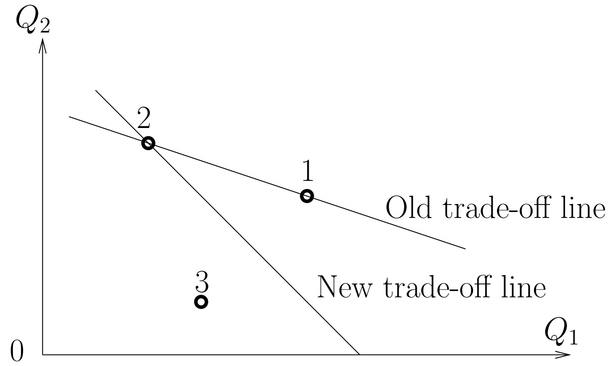


Figure 2: A changed trade-off line resulting from a 'decoy' product, 3.

(new) utility function is just $U(\boldsymbol{Q}) = (Q_1 + Q_2)/\sqrt{2}$. The long-term market shares corresponding to this are 16/29, 9/29 and 4/29. A little has been gained by the first product, and more lost by the second. These changes would be increased if α_{ij} were $(U(\boldsymbol{Q}_i)/U(\boldsymbol{Q}_j))^\gamma$ with $\gamma > 1$.

[4]Taking a power, $\alpha_{ij} = (d(\boldsymbol{Q}_j, \overline{\boldsymbol{Q}})/d(\boldsymbol{Q}_i, \overline{\boldsymbol{Q}}))^\gamma$, could also make things a bit better, for $0 < \gamma < 1$, or accentuate the effect further, for $\gamma > 1$.

3 A linear ODE model

In this model the strength of flux between brands is determined by perceived brand quality, based upon binary comparisons. The simplest response to such comparisons is an attempt by consumers to minimize the expected regret resulting from any choice, which is what is assumed here. It has previously been shown that the choice rule recognizes the attribute-wise proximity of an alternative to other brands [1], and it is therefore appropriate for preference change to be modelled on the pair-wise ranking of brands in each quality, the simplest perhaps being to assign a positive score to a brand for each successful comparison. Thus consumers attempt to minimize their anticipated regret by opting – on any particular quality – for the safe bet. More sophisticated consumer behaviour, capable of not only ranking brands but discriminating according to the size of proximity gap requires more complex modelling, but may be justified since it appears that subjective attribute valuations at least are nonlinear, reference-point-dependent functions [1].

3.1 Consumer preference

Consider a consumer whose preference is shared out amongst all the available brands in a market where there are no null brands, so that the total of all brands' preference shares is 1 (100%). The proportion of consumer preference held by brand X at time t is denoted by $X(t)$. For example, if the preference share of brand A is plotted against brand B in a market where only two brands exists, the point must lie somewhere along the straight line $B(t) = 1 - A(t)$. Of interest is the case when a third brand is added, possibly as a decoy. The additional brand means that the preference distribution changes from being a straight line in the two-dimensional plane (A, B), to a plane in three-dimensional space (A, B, C).

3.1.1 Switching between brands

Consider a linear flux α_{XY} of preference moving to brand X from brand Y. It is assumed that all fluxes are strictly positive, how ever small.[5] Flux is the proportion of consumer preference in one brand which is moving towards another brand, and is distinct from the market share each brand attains (see Subsection 3.1.2). In a two-brand market the resultant differential equations are:

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -\alpha_{BA}A + \alpha_{AB}B\,, \\ \frac{\mathrm{d}B}{\mathrm{d}t} &= +\alpha_{BA}A - \alpha_{AB}B\,, \end{aligned} \tag{10}$$

together with

$$A(t) + B(t) = 1\,. \tag{11}$$

[5]It is inappropriate to consider negative α_{XY} since this would be equivalent to a positive flux α_{YX}, so all that is really being excluded is zero flux.

Note that the system is over-determined: only one of equations (10) together with (11) is required to determine the behaviour of the system. Upon introduction of a third brand into the market, the system will be governed by

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -\left(\alpha_{BA}+\alpha_{CA}\right)A+\alpha_{AB}B+\alpha_{AC}C \\ \frac{\mathrm{d}B}{\mathrm{d}t} &= +\alpha_{BA}A-\left(\alpha_{AB}+\alpha_{CB}\right)B+\alpha_{BC}C \\ \frac{\mathrm{d}C}{\mathrm{d}t} &= +\alpha_{CA}A+\alpha_{CB}B-\left(\alpha_{AC}+\alpha_{BC}\right)C \end{aligned} \tag{12}$$

together with

$$A(t)+B(t)+C(t)=1\,, \tag{13}$$

where again only two out of the three equations (12) are required with (13) to determine the full solution. The model could be reformulated in terms of a single lumped own brand (AC) and a competitor brand (B), but this would obscure any dynamics such as the decoy effect and be inappropriate when attempting to model the preference fluxes between individual brands, which would incorporate the different locations of each in quality space (see Subsection 3.1.4).

3.1.2 Market share

It may be shown that, for non-zero initial conditions, all solutions of (10), (11), converge to the equilibrium

$$(\bar{A},\bar{B})=\left(\frac{\alpha_{AB}}{\alpha_{AB}+\alpha_{BA}},\frac{\alpha_{BA}}{\alpha_{AB}+\alpha_{BA}}\right). \tag{14}$$

Since $(\bar{A},\bar{B})$ is a globally attractive, stable equilibrium, it can be considered as representing the market share of each brand. This is independent of transient changes in preferences and the result of consumer preferences being expressed through purchases.

The system (12), (13) also converges to a globally attractive, stable equilibrium giving the market share of each brand as:

$$\begin{aligned} \left(\hat{A},\hat{B},\hat{C}\right) = \Big(& \left(\alpha_{AC}\alpha_{CB}+\alpha_{AB}\alpha_{BC}+\alpha_{AC}\alpha_{AB}\right)/S_\alpha, \\ & \left(\alpha_{BA}\alpha_{AC}+\alpha_{BC}\alpha_{CA}+\alpha_{BA}\alpha_{BC}\right)/S_\alpha, \\ & \left(\alpha_{CB}\alpha_{BA}+\alpha_{CA}\alpha_{AB}+\alpha_{CA}\alpha_{CB}\right)/S_\alpha\Big), \end{aligned} \tag{15}$$

where

$$\begin{aligned} S_\alpha = \; & \alpha_{AC}\alpha_{CB}+\alpha_{AB}\alpha_{BC}+\alpha_{AC}\alpha_{AB} + \alpha_{BA}\alpha_{AC}+ \\ & \alpha_{BC}\alpha_{CA}+\alpha_{BA}\alpha_{BC} + \alpha_{CB}\alpha_{BA}+\alpha_{CA}\alpha_{AB}+\alpha_{CA}\alpha_{CB}\,. \end{aligned}$$

Here the flux constants α_{XY} represent the decision-making process (see Subsection 3.1.4) with the steady state being the long-term outcome, namely the proportion of each brand actually purchased. Thus the model allows for significant preference flux between brands (large α_{XY}) while market shares ($\hat{X}$) may remain constant. In the context of decoy behaviour, there may be a large flux towards brand C (given by α_{CA} and α_{CB}) without necessarily resulting in a significant market share $\hat{C}$.

3.1.3 Success of the new product as a decoy

The assumption that consumers' preference shares of all the available brands always add to 1 implies that no preference is withheld *e.g.* in expectation of a currently unavailable brand. Thus the total market size is independent of the number of brands, and new brands are not capable of introducing new consumers, *i.e.* $\bar{A} + \bar{B} = \hat{A} + \hat{B} + \hat{C} = \tilde{A} + \tilde{B} + \tilde{C} + \tilde{D} = \ldots = 1$. To improve sales of the target brand A through the introduction of a third brand C requires $\hat{A} > \bar{A}$, which is satisfied, using (14) and (15), if and only if

$$(16) \qquad \alpha_{AC}\alpha_{CB}\alpha_{BA} > \alpha_{CB}\alpha_{BA}\alpha_{AB} + \alpha_{AB}\alpha_{BC}\alpha_{CA} + \alpha_{CA}\alpha_{AB}\alpha_{CB} + \alpha_{CA}\alpha_{AB}^2 \,.$$

To increase overall market share of own brands ($A + C$) requires a reduction in the market share of the competitor brand so that $\hat{B} < \bar{B}$, which is satisfied if and only if:

$$(17) \qquad \alpha_{AB}\alpha_{BC}\alpha_{CA} < \alpha_{AC}\alpha_{BA}\alpha_{CA} + \alpha_{BA}\alpha_{AC}\alpha_{CB} + \alpha_{CA}\alpha_{AB}\alpha_{BA} + \alpha_{CB}\alpha_{BA}^2 .$$

Before considering what form of flux constant might be appropriate, note that if α_{AC} is large compared to all other constants then

$$(18) \qquad \hat{A} \sim \frac{\alpha_{AB} + \alpha_{CB}}{\alpha_{AB} + \alpha_{BA} + \alpha_{CB}} > \frac{\alpha_{AB}}{\alpha_{AB} + \alpha_{BA}} = \bar{A}, \quad \left(\Rightarrow \hat{B} < \bar{B}\right).$$

Thus if there is sufficiently strong change in preference from C to A, then the desired increase in market share A will always occur. How significant this increase is will depend, as can be seen from (18), entirely upon the size of α_{CB}. This intuitively makes sense, since a reasonable preference change from B to C is required for the flux from C to A to produce a significant increase in A.

3.1.4 Brand preference flux

Here preference flux based upon differing brands' relative perceived quality is modelled with two quality measures P and Q. The classic scenario when two brands (A and B) trade off successfully – each outranking the other in exactly one quality dimension – is shown in Fig. 3. A trade-off line exists through such brands, so that, after rescaling, the sum of *perceived* quality values for all brands on the line is equal. New brands below the line should not compete very successfully, while those that come into the market place above the line are expected to fare well. If we denote P_X and Q_X as the two quality values of brand X, then $P_X + Q_X$ is the same for all brands on the trade-off line.

The location of any two existing brands generates up to nine zones into which any new potential brand could be placed. Fewer than nine zones will only exist in the degenerate

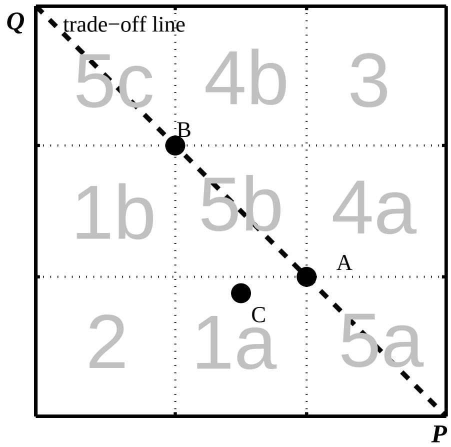


Figure 3: Two brands (A and B) located on a trade-off line relative to two quality dimensions P and Q will generate nine zones (here labelled 1a to 5c) into which any new produce (C) could be added. Here C has been placed in zone 1a, the target-decoy position for A (see Table 1).

case where the two brands are exactly equal in one or more quality dimension, which is not of interest here. If brand A is the target brand with a trade-off competitor B, and brand C is a new (potential decoy) brand, then the nine zones labelled in Fig. 3 are conventionally defined as in Table 1.

5c : trade-off	4b : reverse-competitor-decoy	3 : utopia
1b : competitor-decoy	5b : trade-off	4a : reverse-target-decoy
2 : worthless	1a : target-decoy	5a : trade-off

Table 1: Conventional definitions of the zones in quality space given in Fig. 3.

The simplest outcome of a binary comparisons by a consumer in such a process is to rank two brands as 'better' or 'worse' in each quality. This may be considered as a minimized regret approach, whereby consumers rank the potential for reducing disappointment in any choice over finding the best brand. The simplest reasonable flux constant is therefore given by

$$\alpha_{XY} \; = \; \mathrm{H}\,(P_X - P_Y) + \mathrm{H}\,(Q_X - Q_Y), \tag{19}$$

where H is again the Heaviside function. Thus any brand will independently gain a score when it compares favourably on any individual quality dimension. To allow for consumers with a preference with regard to qualities, perhaps considering one more important than the other, it is appropriate to weight the scores gained from each comparison. Alternatively these weights represent a confidence weighting, given by consumers who value all qualities equally, to their ability to correctly judge the ranking in each quality. In addition, consumers might give extra weighting to any brand which completely dominates another. Such behaviour will result in fluxes of the form:

$$\alpha_{XY} = \beta \mathrm{H}\left(P_X - P_Y\right) + \gamma \mathrm{H}\left(Q_X - Q_Y\right) + \delta \mathrm{H}\left(P_X - P_Y\right) \mathrm{H}\left(Q_X - Q_Y\right), \tag{20}$$

where β, γ and δ are non-negative.

3.2 Results

In order to see what are the minimal requirements to produce behaviour such as the decoy effect it is appropriate to first consider the least complex case. For consumers who only rank qualities without considering the size of any discrepancy or dominance, the flux between brands is given by (19).

Consider a new brand placed in the target-decoy zone 1a (see Fig. 3). Then the flux from this new brand C to A is determined by $\alpha_{AC} = 2$, because it is dominated, but the flux to B has only $\alpha_{BC} = 1$, since B outranks C in Q but not P (consequently the flux from B to C has $\alpha_{CB} = 1$ also). The dynamics of the system are given by substituting each of these values for (19) into (12):

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \beta \begin{pmatrix} -1 & 1 & 2 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix},$$

resulting in a steady-state solution, given by substituting (19) into (15), of

$$\left(\hat{A}, \hat{B}, \hat{C}\right) = \left(\frac{5}{9}, \frac{1}{3}, \frac{1}{9}\right). \tag{21}$$

The original market share of brands A and B was $\bar{A} = \bar{B} = 1/2$, so that market share of A has increased by 1/18 to $\hat{A} = 5/9$. Brand B's market share is reduced by 1/6 to $\hat{B} = 1/3$, lost partly to A and partly to C. The results for all other zones are given in Fig. 4, and show the expected behaviour. Only the target-decoy position (zone 1a) will increase the target brand's market share; it follows that placing a new brand in the competitor-decoy zone (1b) will actually harm overall market position. A worthless brand (zone 2) which is dominated by all others will not gain any market share.

All other locations result in an overall increase in market share between the two own brands A and C, although the only significant gain that can be achieved is if the new brand dominates the competitor (zones 3 and 4b). In all other cases the market share gained is simply equivalent to introducing a third trade-off brand (with $\hat{B}$ remaining at 1/3 in all cases), and so there is no benefit from producing a brand which outranks only the target brand in any qualities. This may easily be seen by clustering brands A and C – see above. These crude rules will obviously be tempered by any weightings attached to qualities or the magnitude of differences, giving more subtle and realistic behaviour.

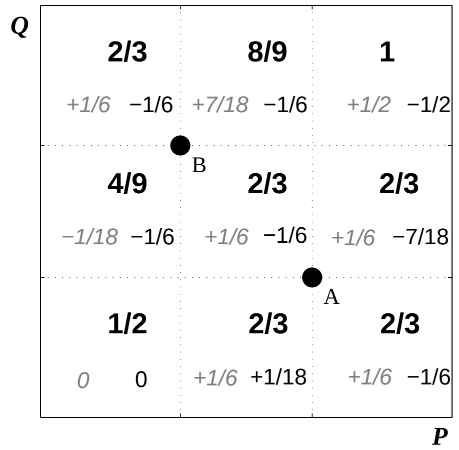


Figure 4: The results of placing brand C into each of the nine possible zones created by A and B. Total market share of own brands (target A plus new brand C), $\hat{A} + \hat{C}$, is given in bold; the net gain in market share, $\hat{A}+\hat{C}-\bar{A}$, is given in italics; the net gain for the target brand, $\hat{A} - \bar{A}$, is given in normal font.

3.2.1 Quality preferences and dominance weighting

Any difference in quality significance, $\beta \neq \gamma$ in (20), may skew the outcome of the consumer preference dynamics. As one quality starts to dominate, the problem tends to a comparison in a single quality dimension. In the limiting case the brand with the highest perceived value in that quality will gain the entire market, since it is considered to completely dominate all other brands. If a consumer's preference change is strengthened by noting that a brand is completely dominated, additional weighting is given by $\delta \neq 0$ in (20). In the absence of quality preference ($\beta = \gamma = \delta$) this results in quantitative but no qualitative differences. For example, the target-decoy position (C placed in zone 1a) results in a market share distribution of

$$\left(\hat{A}, \hat{B}, \hat{C}\right) = \left(\frac{7}{12}, \frac{1}{3}, \frac{1}{12}\right) ,$$

as compared to that given in (21). As expected, such weighting increases the decoy effect.

3.2.2 Multiple brands and additional quality dimensions

The analysis can easily be extended to multiple brands. Note that for only two brands which trade off (always the case where one does not dominate the other) we may draw a straight line through both and define this as the trade-off line. For multiple brands which trade-off, the line may not be straight but will still be monotonic: thus none of the results are affected.

Although consumers may only carry out binary comparisons, these may be executed over more than two quality dimensions. The model extends naturally to higher dimensions in quality-space: the trade-off line in two dimensions becomes a plane in three dimensions, given by $P + Q + R = 1$, upon which equal brands would be expected to lie. For more than two brands to all trade off successfully, however, all brands must trade off pairwise.

3.3 Conclusions

The asymmetric decoy effect may be replicated with minimal prior assumptions, based only upon the aim of minimized regret. Binary comparisons of products on separate quality dimensions are sufficient to drive consumer preference towards a target brand, producing a shift in market share.

If the rate of change from an inferior decoy to a target is sufficiently strong then the desired increase in market share of the target (and consequently the loss in market share of the competitor) will always occur. Furthermore, with such simple strategies only the target-decoy position will increase the target brand's market share, with analogous results for the competitor-decoy zone.

For simple fluxes the best strategy is for the target-dominated decoy to outrank the competitor in all the qualities in which the target outranks the competitor. If consumers place additional significance on a brand dominating another beyond the fact that it outranks the other on each quality dimension separately, as might be expected in a minimzed regret approach attempting to find a 'safe bet', then the size of the decoy effect is increased. Results may be extended to a market place with multiple brands where consumers evaluate these on multiple quality dimensions. At present there appears to be an absence of experimental research considering the decoy effect either for more than three brands or where choices must be made across more than two quality dimensions, [2].

While the model presented here is clearly not sufficient to explain all the many subtleties of consumer choice behaviour, it shows how complex outcomes may result from quite simple driving forces. This suggests that some current models may call for the application of 'Ockham's razor', since it is hard to judge the value of apparent improvements in psychological or sociological modelling unless their inclusion brings about a genuine difference in behaviour or outcome.

4 Sociology: influence of what is being bought

The simplest model considered at the Study Group to account for 'sociology', *i.e.* how one consumer influences another, was an extension of the previous deterministic population model, with just two equally good products on the market. Taking $X(t)$ to be the market share of one of these, the model in the absence of sociological influences would be of the type

$$\frac{\mathrm{d}X}{\mathrm{d}t} = (1 - X) - X = 1 - 2X\,, \tag{22}$$

where time has been scaled to make $\alpha = 1$.

The sociology appears by having another mechanism for brand switching, in addition to the consideration of quality. Here we suppose that people are more likely to buy, or change to, a product if others are doing so. The probability of a single person changing in some small time interval from the second product to the first should be some increasing function of the number of people presently buying the first, *i.e.* an increasing function of the first's market share. Taking this function just to be a power, with exponent $\gamma > 0$, and multiplying by the number susceptible to such a change, this gives a rate of increase of the market share X (additional to the $(1 - 2X)$ in (22)) of $KX^{\gamma}(1 - X)$ for some constant K. (This constant would be negative if people aimed to be different. In this case the new term should probably be $KX^{\gamma+1}$ as it is now a loss term from a population of size X.) There will be a corresponding loss term from the first to the second product, $(1 - X)^{\gamma}X$ (still maintaining no intrinsic bias). The ODE model is now

$$\frac{\mathrm{d}X}{\mathrm{d}t} = 1 - 2X + K(X^{\gamma}(1 - X) - (1 - X)^{\gamma}X)\,. \tag{23}$$

The size of the constant K can be thought of as a measure of the importance of sociology. (The new term for people trying to be different, $K(X^{\gamma-1} - (1 - X)^{\gamma-1})$ with $K < 0$, is expected to lead to similar qualitative behaviour and is not discussed further here.) Having the rate of movement from the second to the first product given by an attractive function of X less a corresponding function of $(1 - X)$, if it is thought that people are less likely to change *from* a popular product, would lead to $(X^{\gamma} - (1 - X)^{\gamma})(1 - X)$ in place of $X^{\gamma}(1 - X)$. This supposed positive term becomes negative for $X < 1/2$ so this idea has not been pursued.)

If $\gamma = 1$, the social influences are linear, but this leads back to (22) as the new terms cancel out. One might also want to avoid this sort of dependency because it would make (23) quadratic, which would conflict with the symmetry that should be inherent in a supposedly unbiased model. Taking instead $\gamma = 2$, the right-hand side of (23) becomes cubic:

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= 1 - 2X + K(X^2(1 - X) - (1 - X)^2X) \\ &= 1 - 2X + KX(1 - X)(2X - 1) = (1 - 2X)(1 - KX + KX^2)\,. \end{aligned} \tag{24}$$

For $K \leq 4$ ('weak' sociology), $1 - KX + KX^2 \geq 0$ and the only steady state is the obvious symmetric one of $X_S = 1/2$; this is stable and a global attractor. For $K > 4$ ('strong' sociology), $1 - KX + KX^2 = 0$ has roots $X_\pm = (K \pm \sqrt{K^2 - 4K})/2K$. 'Pitchfork' bifurcation occurs at $K = 4$. In this case X_S is unstable while $X_\pm$ are stable: 'lock-in' occurs with one product winning the majority of custom despite being no better than the other. Clearly, for K only marginally bigger than 4 these two new equilibria are near the original one but for large K they correspond to market dominance by one product or the other: $X_- \to 0$ and $X_+ \to 1$ as $K \to \infty$.

It is also easy to see that even with one product being transparently better than the other, so that (24) might be replaced by

$$\frac{\mathrm{d}X}{\mathrm{d}t} = 1 - \alpha X + KX(1 - X)(2X - 1)$$

for some α between 1 and 2 if the first product 'should' gain the larger market share, lock-in again occurs. With sufficiently large K there are again three steady states, the smallest and largest of which are stable, and these two will lie close to 0 and 1 if K is very large. In these circumstances, it is possible for the inferior product to hold on to nearly all the market.

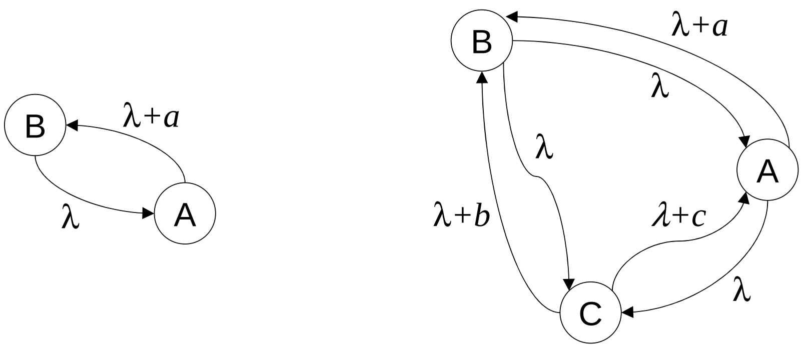


Figure 5: A Markov Chain for (left) two products and (right) three products. The transition probabilities between products at each time step are marked with arrows; the (unmarked) probabilities of remaining with the current product can be deduced.

5 A Markov model with social influences

In this section we present a model based on Markov chains, rather than the continuous-time differential equations considered so far. We first develop results for the possibility of a decoy effect, similar to those found in Section 3. We then introduce sociology and obtain results for lock-in analogous to those found in Section 4. These Markov models display both important similarities to and differences from the previous models, and may be simpler to work with and visualise in certain circumstances.

5.1 The decoy effect in a Markov model

We start by considering a simple probabilistic model without sociology, with the intention of examining the possibility of a decoy product. Consider first two products A and B as in Fig. 5 (left): the transition probabilities shown between products produce, in the notation of (1) in Section 1.5, a transition matrix (α^*_{ij}) given by

$$P = \begin{pmatrix} 1-\lambda-a & \lambda \\ \lambda+a & 1-\lambda \end{pmatrix}.$$

Here λ represents a natural 'churn rate', that is, the effect of a consumer changing product simply because he or she is unable to make a rational choice between the two. The constant a represents a bias which the consumer may feel to leave product A for B, perhaps because B is of higher overall quality (that is, offers a higher value for the consumer's utility function). The equilibrium distribution $\boldsymbol{\pi}$ (that is, the vector of equilibrium probabilities such that $P\boldsymbol{\pi} = \boldsymbol{\pi}$) is found to be

$$\boldsymbol{\pi} = \left(\frac{\lambda}{2\lambda+a}, \frac{\lambda+a}{2\lambda+a} \right)^T.$$

Introducing now a third product C as in Fig. 5 (right), with the same churn rate[6] but with additional biases b and c as shown, we obtain a transition matrix

$$P_3 = \begin{pmatrix} 1-2\lambda-a & \lambda & \lambda+c \\ \lambda+a & 1-2\lambda & \lambda+b \\ \lambda & \lambda & 1-2\lambda-b-c \end{pmatrix}$$

with equilibrium distribution

$$\boldsymbol{\pi} = \left(\frac{\lambda(3\lambda+b+2c)}{(3\lambda+a)(3\lambda+b+c)}, \frac{3\lambda^2+\lambda(2a+2b+c)+a(b+c)}{(3\lambda+a)(3\lambda+a+c)}, \frac{\lambda}{3\lambda+b+c} \right)^T.$$

In order for a decoy effect to be exhibited, we wish for the market share of product A to be increased in the presence of the decoy product C, that is,

$$\frac{\lambda(3\lambda+b+2c)}{(3\lambda+a)(3\lambda+b+c)} > \frac{\lambda}{2\lambda+a}$$

or

$$\alpha\gamma - \beta + \gamma > 3 \tag{25}$$

where

$$\alpha = a/\lambda, \quad \beta = b/\lambda, \quad \gamma = c/\lambda.$$

We also require

$$\gamma > \beta > \alpha \geq 0 \tag{26}$$

in order that the initial market share of A is lower than B; that the bias from the decoy to B is stronger than the bias from A to B; and that the bias from the decoy to A is stronger still.

Equations (25) and (26) together therefore define the region of interest in (α, β, γ)-space. The fractional increase in the market share of product A through the introduction of the decoy product is

$$\frac{(2+\alpha)(3+\beta+2\gamma)}{(3+\alpha)(3+\beta+\gamma)}.$$

[6]It is debatable whether the *same* churn rate λ should be used in the three-product model as in the two-product model. Clearly, introducing a third product will increase the level of churn, as consumers have more choices available. However, if the same value of λ is used for both models then that implies that the probability of leaving A for another product is roughly doubled simply by the introduction of an extra, inferior product. A new value for λ lying between the old value and half of that might be most appropriate, and that would change the results obtained later in this section. The correct relationship between the values of λ in models can only be determined by measuring how consumers react to additional choice in practice.

Note that in order for all the probabilities in the 3-product transition matrix P_3 to remain in the range $[0,1]$, we must require that both $1-2\lambda-a$ and $1-2\lambda-b-c$ are non-negative. For given values of α, β and γ satisfying (26), this is equivalent to

$$\lambda \leq \frac{1}{2+\beta+\gamma},$$

which sets a maximum value on the churn rate. The decoy effect is best observed when γ is made large (a high bias from C to A and/or a low churn rate) whilst at the same time limiting the size of β.

5.1.1 Special case: equal initial market shares

If the two products A and B are initially equally placed in the market, then we take $a=0$ so that each has a 50% market share before the introduction of the decoy product. Then (25) and (26) reduce to the requirement that

$$\gamma > 3+\beta.$$

This is likely to be rather difficult to achieve in practice, as it requires making the bias towards A significantly larger than the bias towards B, unless the churn rate λ is small (in which case b and c could both be small as well but differ by more than 3λ). The fractional increase in the market share of A in this case would be

$$\frac{2(3+\beta+2\gamma)}{3(3+\beta+\gamma)} \leq \frac{4}{3}.$$

If β and γ are large but of similar size then the increase in market share is negligible; hence simply taking λ small is not on its own sufficient for an appreciable decoy effect, and more care must be taken in engineering appropriate values.

5.2 Sociology in a two-product Markov model

We now return to just two products, as in Fig. 5 (left), but introduce a sociological effect whereby consumers have some tendency to prefer a product which other people are already buying. Suppose that we have a total of N consumers, n of whom at a particular time-step are buying product A and the remaining $N-n$ buying product B. The sociological effect is modelled by changing the transition probability for $B \to A$ from λ to

$$\lambda + \mu_1 n/N, \tag{27}$$

and the transition probability for $A \to B$ from $\lambda + a$ to

$$\lambda + a + \mu_2 (N-n)/N, \tag{28}$$

where μ_1 and μ_2 are constants representing the strength of the sociology (analogous to K in Section 4). In a more advanced model, μ_1 and μ_2 might vary from consumer to consumer.

Each of the N consumers follows his or her own Markov chain, and the value of n changes whenever a consumer switches product. There is therefore a related Markov chain for the value of n, with states $\{0, 1, \ldots, N\}$, and we consider the equilibrium distribution $\boldsymbol{\pi}$ for this new Markov chain ($\boldsymbol{\pi} \in \mathbb{R}^{N+1}$).

We now use a standard argument from the theory of stochastic processes, in which the transition probabilities above are to be considered as *rates* of transition per unit time, and consider a time interval δt. The probability that n increases by one during this time interval is $(N-n)(\lambda + \mu_1 n/N)\delta t$, because there are $N-n$ consumers currently buying product B, each of whom might switch to A during the interval with probability $(\lambda+\mu_1 n/N)\delta t$. We take δt sufficiently small that the likelihood of two or more consumers switching simultaneously during the time interval is negligible. Similarly, the probability that n decreases by one is $n\big(\lambda + a + \mu_2(N-n)/N\big)\delta t$.

At equilibrium, there must be a balance between the numbers flowing from state n to $n+1$ and vice versa. Therefore,

$$\pi_n(N-n)\left(\lambda + \frac{\mu_1 n}{N}\right) = \pi_{n+1}(n+1)\left(\lambda + a + \frac{\mu_2(N-n-1)}{N}\right). \tag{29}$$

This recurrence relation allows each of the π_n to be determined in terms of π_0, which is then found (if required) from the normalisation condition $\sum_n \pi_n = 1$. The expected market share of product A is given by $\sum_n n\pi_n$.

5.2.1 Analytic results

While (29) provides an analytic result, it is difficult to get a clear idea of the distribution. Further progress can be made by considering a continuum model for large N.

Let $x = n/N$ and consider the function $\phi(x) = N^{-1}\ln(\pi_{Nx})$. Initially, ϕ is only defined on the set $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\}$ (*i.e.* when Nx is an integer), but we can extend this to a continuum approximation to ϕ defined for $x \in [0,1]$. From (29),

$$\mathrm{e}^{N\phi(x)}(N-n)(\lambda + \mu_1 n/N) = \mathrm{e}^{N\phi(x+1/N)}(n+1)(\lambda + a + \mu_2(N-n-1)/N).$$

Replacing $\phi(x+1/N)$ with the Taylor approximation $\phi(x) + N^{-1}\phi'(x)$ we obtain

$$\mathrm{e}^{\phi'(x)} = \frac{(1-x)(\lambda+\mu_1 x)}{(x+1/N)(\lambda + a + \mu_2(1-x-1/N))},$$

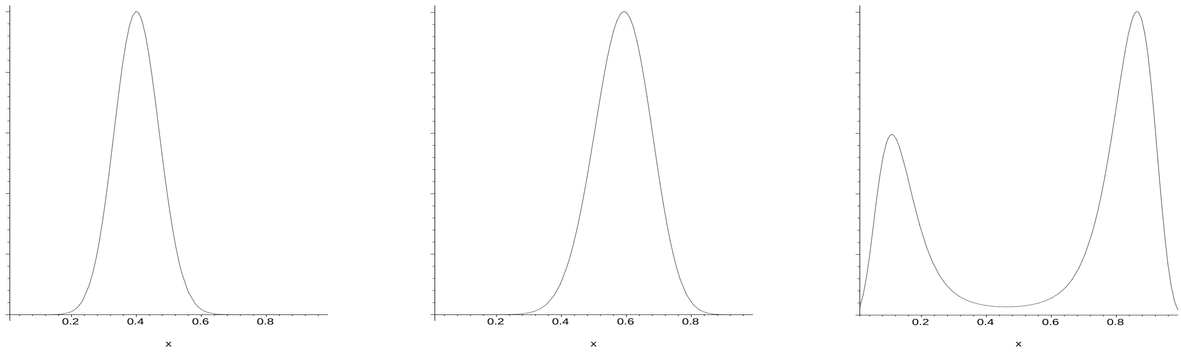


Figure 6: Graphs of the equilibrium distribution π_{Nx} for $x \in [0,1]$ (vertical scale arbitrary), given by the approximation $\exp\{N\phi(x)\}$, showing the long-term distribution of the proportion of consumers purchasing product A. In each case, there are 50 consumers ($N = 50$), a fairly small churn rate ($\lambda = 0.1$) and a bias $a = 0.05$, so that product B has a small natural advantage (because $a > 0$). Left: the linear model (30) with no sociological effect (*i.e.* $\mu_1 = \mu_2 = 0$). The expected market share of product A is calculated to be 40%, which agrees with the value $\lambda/(2\lambda+a)$ given in Section 5.1. Centre: the linear model with $\mu_1 = 0.3$, $\mu_2 = 0.1$, so that both products enjoy a sociological effect but product A has stronger 'bonding' (the number of people buying A has a greater effect on other consumers than those buying B). The expected market share is calculated to be 59%, so the brand managers of product A have effectively countered the natural advantage of product B using sociology. Right: the quadratic model (33) with $\mu_1 = 1.3$, $\mu_2 = 1$. In this case the marketplace becomes strongly polarised, and consumers clearly 'lock-in' to just one of the products. The two peaks near $x = 0$ and 1 show that the most likely outcomes are that almost all consumers buy product A, or almost all buy product B: which of the two actually occurs will depend on random initial fluctuations (and it is possible that after a long time the situation might randomly 'flip' to the other peak). The concept of 'expected market share' is effectively meaningless here.

or, in the limit $N \to \infty$ (and nonzero x),

$$\phi'(x) = \ln \frac{(1-x)(\lambda + \mu_1 x)}{x(\lambda + a + \mu_2(1-x))}. \tag{30}$$

This equation may be solved exactly, though the solution is too messy to be included here. The solution for ϕ represents the long-term distribution of the proportion of people buying product A.

The expected market share of A is easily shown to be

$$\frac{\int_0^1 x \exp\bigl(N\phi(x)\bigr)\,\mathrm{d}x}{\int_0^1 \exp\bigl(N\phi(x)\bigr)\,\mathrm{d}x}.$$

For large N, this is given approximately by the value of x at which ϕ has a maximum; this value can easily be found by solving $\phi'(x) = 0$ analytically. If $\mu_1 = \mu_2$ (*i.e.* the sociological impacts of A and B are identical) then the market share is found to be $\lambda/(2\lambda + a)$, which is, as expected, identical to the market share without sociology found above in Section 5.1. However, if A and B have different sociological impacts then the market share may be affected significantly: see Fig. 6 (left and centre).

This Markov model is a linear one, in the sense that sociological effects are modelled as linear functions of n in (27) and (28). The corresponding ODE model comes from taking $\gamma = 1$ in Section 4.[7] Interesting results can also be obtained with a quadratic Markov model (corresponding to the ODE model with $\gamma = 2$) by changing the transition rates given in (27) and (28) to

$$\lambda + \mu_1 (n/N)^2 \tag{31}$$

and

$$\lambda + a + \mu_2[(N-n)/N]^2 \tag{32}$$

respectively. Much of the above analysis is unaltered; the new equation governing $\phi(x)$ is

$$\phi'(x) = \ln \frac{(1-x)(\lambda + \mu_1 x^2)}{x(\lambda + a + \mu_2(1-x)^2)}, \tag{33}$$

which can also be solved exactly (but even more messily than before!). The solution for ϕ can now be either singly or doubly peaked; the double-peaked solution is akin to the two stable equilibria found in the ODE model of Section 4. In such a solution, 'lock-in' may be clearly indicated, as shown in Fig. 6 (right): either one of the products can capture a significant proportion of the market.

[7] In Section 4, taking $\gamma = 1$ did not induce a change in market share, but this was because the two products were assigned equal sociological impacts.

6 An agent-based model with social influences

In this section, we present a more general phenomenological framework for modelling sociology in the process of choosing a product. This model borrows ideas from the theory of coupled oscillators. Individual buyers are described by their intrinsic inclination to buy one of two products, which we call the *target* and *competitor* products, in keeping with Section 3. In the absence of social interaction with other consumers, the probability $x_i(t)$ for an individual i to buy the target product evolves in time according to

$$\frac{dx_i}{dt} = x_s\left(i\right) - x_i.$$

In this equation, the 'time' t is an evolutionary parameter that may differ from actual physical time; it can, for instance, measure the number of purchases previously made by the consumer. The quantity $x_s\left(i\right)$ represents the 'natural inclination' of consumer i to buy the target product and depends solely on i's psychological profile.

Sociology is built into the model by way of introducing a coupling function $\Gamma\left(x_i, \{x_j\}\right)$ between individual i and other consumers such that

$$\frac{dx_i}{dt} = x_s\left(i\right) - x_i + \Gamma\left(x_i, \{x_j\}\right).$$

The precise form of the coupling function is at the heart of the modelling process. In this work, we construct the simplest possible function that is able to reproduce some of the qualitative collective effects observed in consumer behaviour.

As a first simplifying assumption, we shall postulate that a consumer is affected only by the mean behaviour of the other consumers[8]. Thus, to begin with, we will assume that

$$\Gamma\left(x_i, \{x_j\}\right) = \Gamma\left(x_i, X\right), \qquad X = \frac{1}{N}\sum_{j=1}^{N} x_j,$$

where N is the total number of consumers. The variable X can be interpreted as the market share of the target product. If $X = \frac{1}{2}$, then no clear trend exists in the market and we therefore expect the coupling function to vanish. Hence, we will write

$$\Gamma\left(x_i, X\right) = \tilde{\Gamma}\left(x_i, X\right)\left(X - \frac{1}{2}\right)$$

for some $\tilde{\Gamma}$.

We further assume that a given consumer will be more sensitive to sociological effects if he is uncertain of his own choice, *i.e.* if his current probability of buying the target product is close to $\frac{1}{2}$. To formalize this hypothesis, we set

$$\tilde{\Gamma}\left(x_i, X\right) = \tilde{\Gamma}\left(X\right)\left(\frac{1}{2} - \left|x_i - \frac{1}{2}\right|\right).$$

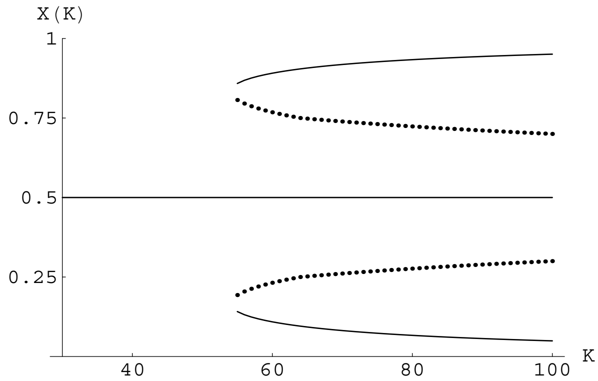


Figure 7: Bifurcation diagram showing all possible states $X(K)$ for $\psi(x_s) = 1$. Full line: stable solution; dotted lines: unstable solution.

Finally, in order to observe some kind of phase transition in the dynamics, the coupling strength has to increase with $\left|X - \frac{1}{2}\right|$. Since this coupling strength must be positive, we assume that

$$\tilde{\Gamma}(X) = K\left(X - \frac{1}{2}\right)^2 + O\left(\left(X - \frac{1}{2}\right)^4\right).$$

Thus, the model we study is

$$\frac{dx_i}{dt} = x_s(i) - x_i + K\left(\frac{1}{2} - \left|x_i - \frac{1}{2}\right|\right)\left(X - \frac{1}{2}\right)^3. \tag{34}$$

The input parameter K is the strength of coupling. We will now examine how the overall share X of the target product evolves as a function of K.

6.1 Analytical results in steady state

In the steady state,

$$x_i = \begin{cases} \dfrac{x_s(i)}{1 - K(X - 1/2)^3} & \text{if } x_i < 1/2 \\[2ex] 1 + \dfrac{x_s(i) - 1}{1 + K(X - 1/2)^3} & \text{if } x_i > 1/2. \end{cases} \tag{35}$$

Accordingly, the psychological profile x_s^* that corresponds to an equiprobable choice, $x_i = 1/2$, for the two products is given by

$$x_s^* = \frac{1}{2}\left[1 - \frac{K}{2}\left(X - \frac{1}{2}\right)^2\right],$$

[8] It would be possible, and probably more realistic, to implement instead a nearest-neighbour coupling, or a 'small-world' network in the function Γ.

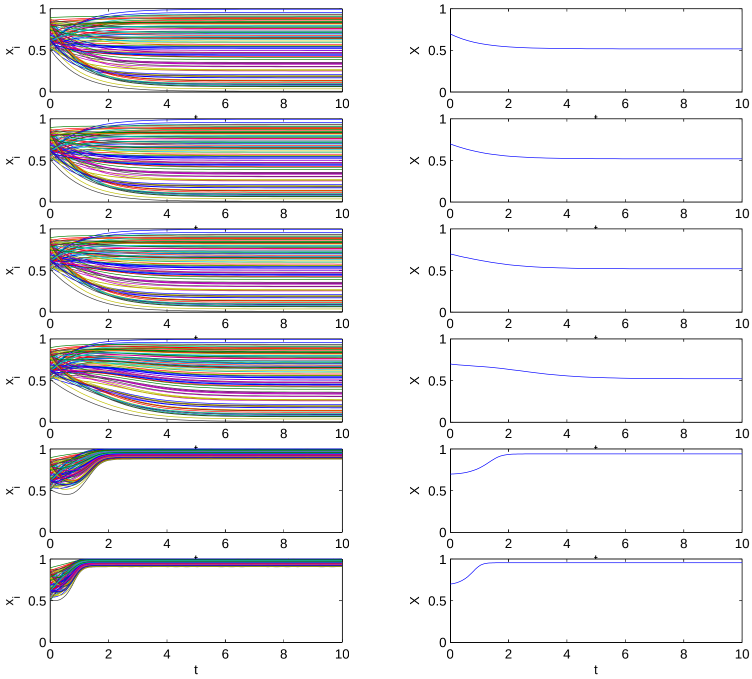


Figure 8: Numerical integration of equations (34), with 100 consumers ($N = 100$). For each variable x_i, a random number $x_s(i)$ was picked from a uniform distribution between 0 and 1. Left: superposition of individual time-traces. Right: evolution of $X(t)$. From top to bottom: $K = 0$, 20, 40, 60, 80, 100.

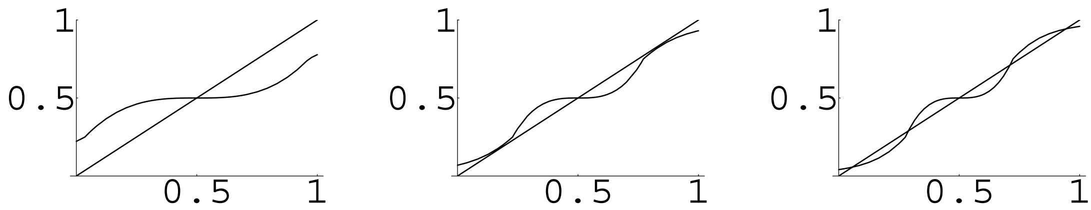


Figure 9: Graphical resolution of the transcendental equation $X = F(X,K)$ with $K = 10$, 50, and 90 from left to right. A uniform distribution of psychological profiles, $\psi(x_s) = 1$, is assumed. As K increases, new solutions appear which exhibit collective behaviour. The 'indecisive' state $X = \frac{1}{2}$ is always a solution.

provided, of course, that the right-hand side is between 0 and 1. Knowing (35), we may now evaluate X :

$$\begin{aligned} X &= \frac{1}{N}\sum_i x_i \\ &= \frac{1}{N}\sum_{x_s(i)<x_s^*}\frac{x_s(i)}{1-K(X-1/2)^3}+\frac{1}{N}\sum_{x_s(i)>x_s^*}\left(1+\frac{x_s(i)-1}{1+K(X-1/2)^3}\right) \\ &= F(X,K)\,. \end{aligned} \tag{36}$$

The value of X is therefore given by the nonlinear equation $X = F(X,K)$. As we will see, for small values of K, the only possible solution is $X = 1/2$. However, new solutions appear for larger values of the coupling parameter K. Let us note that if the number of consumers is very large, then we can evaluate the function $F(X,K)$ in the 'continuum limit':

$$F(X,K) = \int_0^{x_s^*}\frac{\psi(x_s)\,x_s}{1-K(X-1/2)^3}\,\mathrm{d}x_s + \int_{x_s^*}^1 \psi(x_s)\left(1+\frac{x_s-1}{1+K(X-1/2)^3}\right)\mathrm{d}x_s,$$

where we have introduced the probability density $\psi(x_s)$ associated with a given psychology profile.

Our simplifying assumptions for the form of the function Γ allow us to illustrate easily the onset of collective behaviour in the consumption process. Fig. 9 shows the graphical resolution of equation (36). As the coupling strength K increases, new solutions appear where one product is markedly favoured over the other. A bifurcation diagram representing all the possible solutions is given in Fig. 7. One should note that the 'indecisive' state $X = 1/2$ is always a solution, and moreover is always stable. This means that, in the context of our simplified model, a significant perturbation in the market has to be introduced in order to depart from $X = 1/2$, *e.g.* through an advertising campaign. This is illustrated in Fig. 8, where the time evolution of the market is plotted. In the initial conditions, we assumed that one product was slightly more fashionable than the other, although the natural inclination to buy either of the two products, quantified by $x_s(i)$, was uniformly distributed. For sufficiently strong social interactions – large K – a rapid transition to 'locked' state with X close to 1 is observed.

7 The decision process

The standard Logit model for consumer choice assumes that the probability, p_i, with which a consumer buys a given product i from a range of products $1, \ldots, n$ depends on the value the consumer attaches to that product V_i, and the price of the product P_i. This dependence is taken to be of exponential form (thus guaranteeing that all probabilities are positive)

$$p_i = C \exp\left(V_i - sP_i\right),$$

where s is a measure of the relative importance of price to the consumer, and C is a normalisation constant chosen so that

$$\sum_{i=1}^{n} p_i = 1.$$

The value V_i is then taken to reflect the influence on the consumer of the quality of the product, Q_i, the increased likelihood of the consumer buying the same brand as he bought previously (the loyalty effect), and the influence of neighbours (meaning people who have an influence on a consumer, rather than physical neighbours). Each of these dependences is taken to be linear, giving

$$V_i = aQ_i + lI_i + hN_i,$$

where I_i is an indicator function which is unity if the consumer previously bought product i and zero otherwise, N_i is the number of neighbours who bought product i, and a, l and h are constants measuring the relative strength of each effect (termed astuteness, loyalty, and herding respectively).

In such a model the products are all treated independently: the only coupling between the probabilities occurs through the normalisation constant C.

Unilever suggest that there is evidence that consumers do not evaluate all products independently in this way, but in fact make comparisons between products when deciding which to buy. Thus in general the probability of buying product i will depend not only on the value and price of that product, V_i and P_i, but on the value and prices of all products. The goal of this section is to formulate a model for this probability which is based on *binary comparisons*, that is, on (possibly successive) comparisons of two products at a time.

7.1 Binary comparisons

Suppose a consumer chooses to compare product i with product j. What is the probability that he will choose i over j?

In the Logit model we would simply have

$$p_i = C \exp\left(V_i - sP_i\right), \quad p_j = C \exp\left(V_j - sP_j\right), \tag{37}$$

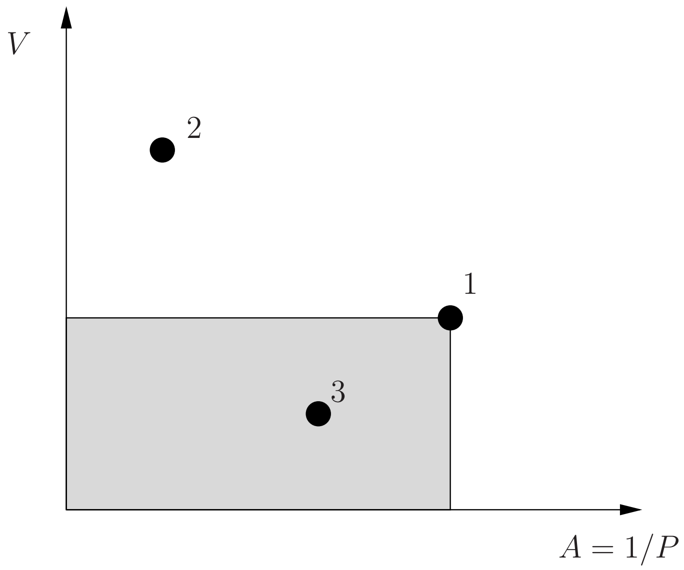


Figure 10: Products in the affordability-value plane. The shaded region is the region of products dominated by product 1.

where C is chosen so that $p_i + p_j = 1$.

An alternative approach would be to suppose that the probability depends on the *difference* in value and the difference in price of the two products, that is

$$p_i = 1 - p_j = F\left((V_i - V_j) - s(P_i - P_j)\right), \tag{38}$$

where F is some function that maps $(-\infty, \infty)$ to $[0, 1]$ (for example $F(x) = (1 + \tanh(x))/2$).

Another approach would be to consider the position of the products in the affordability-value plane (where affordability $A = 1/\text{price}$), and introduce the idea of *dominance*. Consider, for example the three products shown in Fig. 10. If we try and compare first products 1 and 2, we see that product 2 has a lower affordability (i.e. a higher price) than product 1, but it also has a higher value. Thus the consumer has to make a decision on which of value and price is the most important. However, when we compare products 1 and 3, product 3 is of lower value than product 1, and has a lower affordability, and therefore given a choice between product 3 and product 1 a rational consumer should choose product 1 every time. In this situation we say that product 1 dominates product 3. In general a product dominates any product which lies to the left and below it in the affordability-value plane, so that product 1 dominates any product in the shaded region in Fig. 10.

For the comparisons between products where neither is dominated, we can use either the Logit model (37) or the alternative (38).

To see how such a model gives rise to the probabilities of choosing each product we need to construct a decision tree. In forming the decision tree we have to decide which two

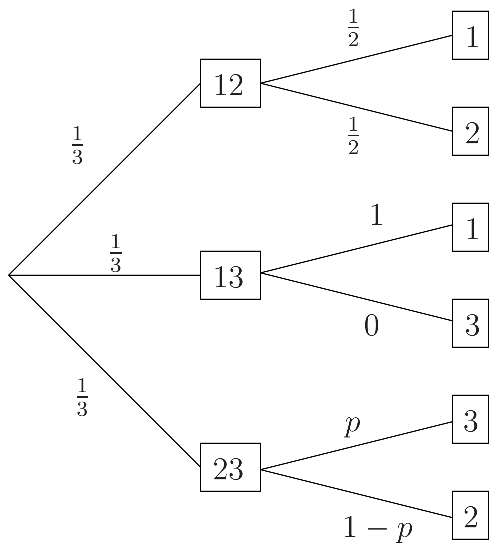


Figure 11: Level 1 decision tree for the products shown in Fig. 10.

products the consumer will choose to compare. We assume initially for simplicity that this decision is uniformly random, so that there is an equal probability of choosing each pair. We also have to decide how many comparisons the consumer will make, that is, having chosen a winner between the first two products, the consumer may then compare this winner with another product, and so on. Each level of the decision tree will have two stages, the choice of which products to compare, and the outcome of the comparison.

7.1.1 Level 1 decision tree

The simplest decision tree for the three products shown in Fig. 10, with one level of comparison (that is, the consumer stops after comparing just two products) is shown in Fig. 11. Let us assume that products 1 and 2 are equally attractive to the consumer, so that in the absence of product 3 each would have a probability of selection of 1/2. We can then see what the introduction of product 3 does to this balance of probabilities, and in particular whether there is a *decoy effect*, that is, since product 1 dominates product 3, the introduction of product 3 might mean that more people will choose to buy product 1 (since product 1 will win in any comparison between the two). In this scenario product 3 acts as a decoy, channelling consumers to product 1. Let us assume that the probability of choosing product 3 over product 2 in a comparison is p.

The probability of reaching any leaf in the decision tree is simply the product of the probabilities of taking each branch required to get there. Thus, after introducing product

3 the probabilities of choosing each product are

$$\begin{aligned} p_1 &= \frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times 1=\frac{1}{2}, \\ p_2 &= \frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times(1-p)=\frac{1}{2}-\frac{p}{3}, \\ p_3 &= \frac{1}{3}\times 0+\frac{1}{3}\times p=\frac{p}{3}. \end{aligned}$$

Thus we see that in this simple model there is no decoy effect; the probability of choosing product 2 has decreased, but the probability of choosing product 1 is the same as it was before product 3 was introduced. This (perhaps surprising) result occurs because although there is now a proportion of binary comparisons that product 1 is guaranteed to win (those between products 1 and 3), there is also a proportion of comparisons that do not involve product 1 at all (those between products 2 and 3). Thus there is a chance that the consumer does not even choose to examine product 1, and this exactly offsets the effect of the decoy.

A simple modification of this model can be used to consider the effect of a decoy on product 1 when there is more than one competitor. Suppose that instead of one competitor (product 2) there are n competitors to product 1, and that all the competitors are equivalent from the point of view of the consumer. Then we can lump all the competitors into a single product number 2, with the main change to the decision tree being to the chance of choosing products to compare. Of course it is now possible that the consumer chooses to compare two competitors products, and we must take this into account. The new decision tree is shown in Fig. 12.

The probabilities of choosing each product are

$$\begin{aligned} p_1 &= \frac{2n}{(n+2)(n+1)}\times\frac{1}{2}+\frac{2}{(n+2)(n+1)}\times 1=\frac{1}{(n+1)}, \\ p_2 &= \frac{2n}{(n+2)(n+1)}\times\frac{1}{2}+\frac{2n}{(n+2)(n+1)}\times(1-p)+\frac{n(n-1)}{(n+2)(n+1)} \\ &= \frac{n}{n+1}-\frac{2np}{(n+2)(n+1)}, \\ p_3 &= \frac{2np}{(n+2)(n+1)}. \end{aligned}$$

Thus, again, we see that there is no decoy effect; product 1 has exactly the same market share as it would have if product 3 were not present.

7.1.2 Level 2 decision tree

Let us now go back to just three products, and assume that the consumer does not stop at one level of comparison, but takes the winner of the first comparison and compares it with the remaining product. In this case we have the level 2 decision tree shown in Fig. 13. Note that for the second comparison there is no choice of products to compare, since there is only one product remaining.

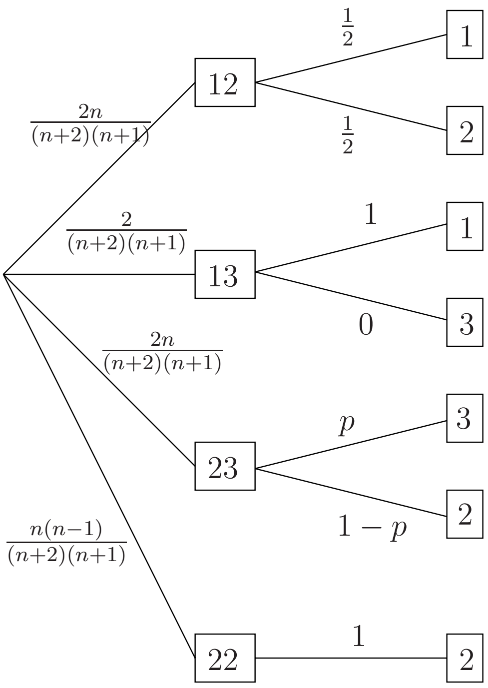


Figure 12: Level 1 decision tree for product 1, n identical competitor products 2, and a decoy product 3.

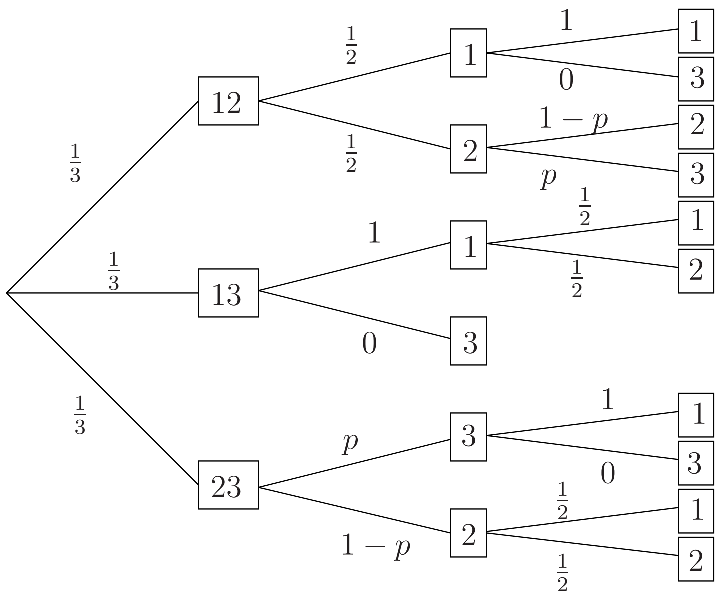


Figure 13: Level 2 decision tree for 3 products.

The probabilities of choosing each product are now

$$\begin{aligned} p_1 &= \frac{1}{6} + \frac{1}{6} + \frac{p}{3} + \frac{1-p}{6} = \frac{1}{2} + \frac{p}{6}, \\ p_2 &= \frac{1-p}{6} + \frac{1}{6} + \frac{1-p}{6} = \frac{1}{2} - \frac{p}{3}, \\ p_3 &= \frac{p}{6}. \end{aligned}$$

Thus in this case the market share of product 1 does increase, so that there is a decoy effect. Note, however, that the market share of the decoy product number 3, is the same as the increased market share of product 1.

7.2 Loyalty

7.2.1 Product awareness

Let us now try and model the effect of product loyalty. We will use the model to determine the probability that the consumer buys product i this time given that he bought product j last time. In the language of Markov chains these are the *transition probabilities* for the state of the system, with the state being given by the last product the consumer bought.

Now, we can model the effect of loyalty as in the Logit model, by increasing the probability of a product winning a binary comparison if it is the product which was bought last time. A more immediate effect of loyalty though, is in our initial choice of products to compare. Rather than the initial choice of two products to compare being random, we will choose to compare our current product with another one chosen at random.

Implementing this idea in our three-product scenario, we now have three decision trees, one for each current choice of product. These are shown in Fig. 14.

If we now let p_{ij} be the probability of moving from product j to product i then the transition matrix $T = (p_{ij})$ is given by

$$T = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} - \frac{p}{2} & \frac{1}{2} - \frac{p}{2} \\ 0 & \frac{p}{2} & \frac{p}{2} \end{pmatrix}.$$

The average market share of each product can now be calculated as a fixed point of the transition matrix, that is, an eigenvector with unit eigenvalue. If f_i is the market share of product i (f_i is average fraction of times that the consumer will buy product i), then

$$T \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

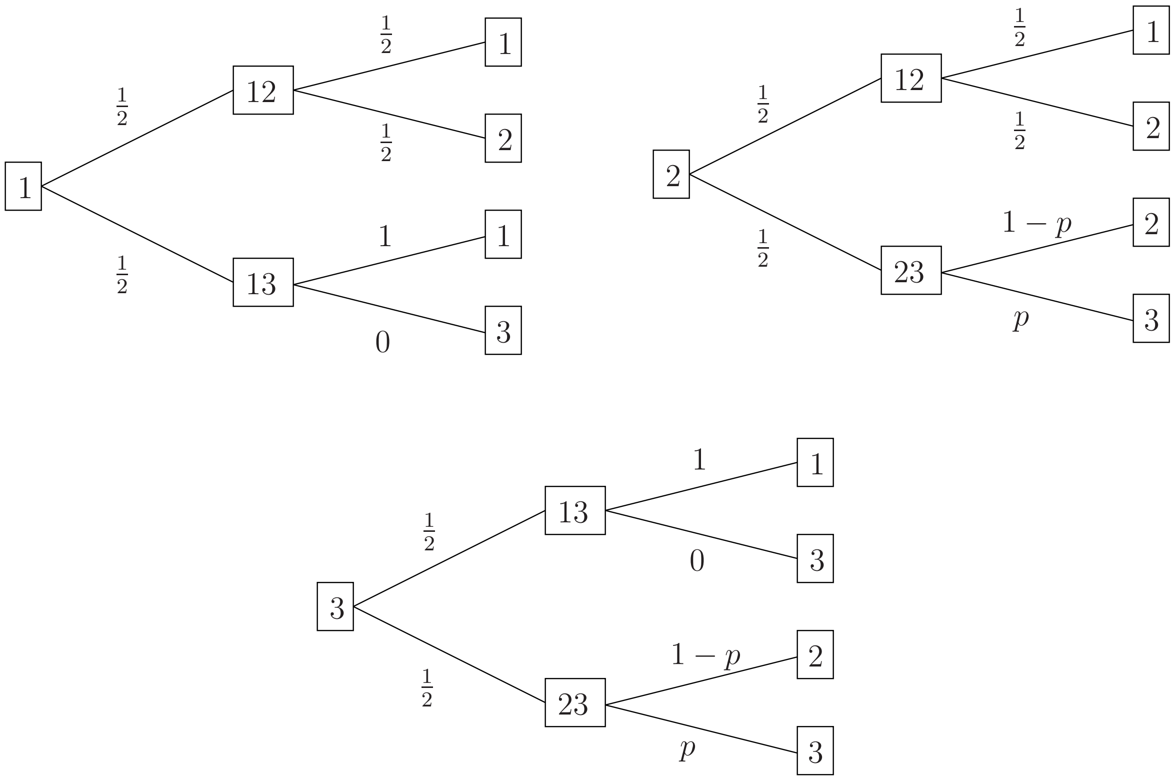


Figure 14: Level 1 decision tree for transition probabilities with 3 products.

giving

$$f_1 = \frac{2+p}{4+p}, \quad f_2 = \frac{2-p}{4+p}, \quad f_3 = \frac{p}{4+p}.$$

For small p these are

$$f_1 \sim \frac{1}{2} + \frac{p}{8}, \quad f_2 \sim \frac{1}{2} - \frac{3p}{8}, \quad f_3 \sim \frac{p}{4}.$$

Thus there is a small decoy effect, but the market share of the decoy product is twice the increase in market share of product 1.

7.2.2 Perceived product value

Let us now also include the effect that loyalty may have on individual comparisons. The model we use model to determine the probability of winning a comparison is a variation of (38). Specifically, we set

$$\text{prob(choose } i \text{ over } j) = F(\Delta_{ij}) \tag{39}$$

where

$$\Delta_{ij} = s_V(V_i - V_j) + s_A(A_i - A_j) \tag{40}$$

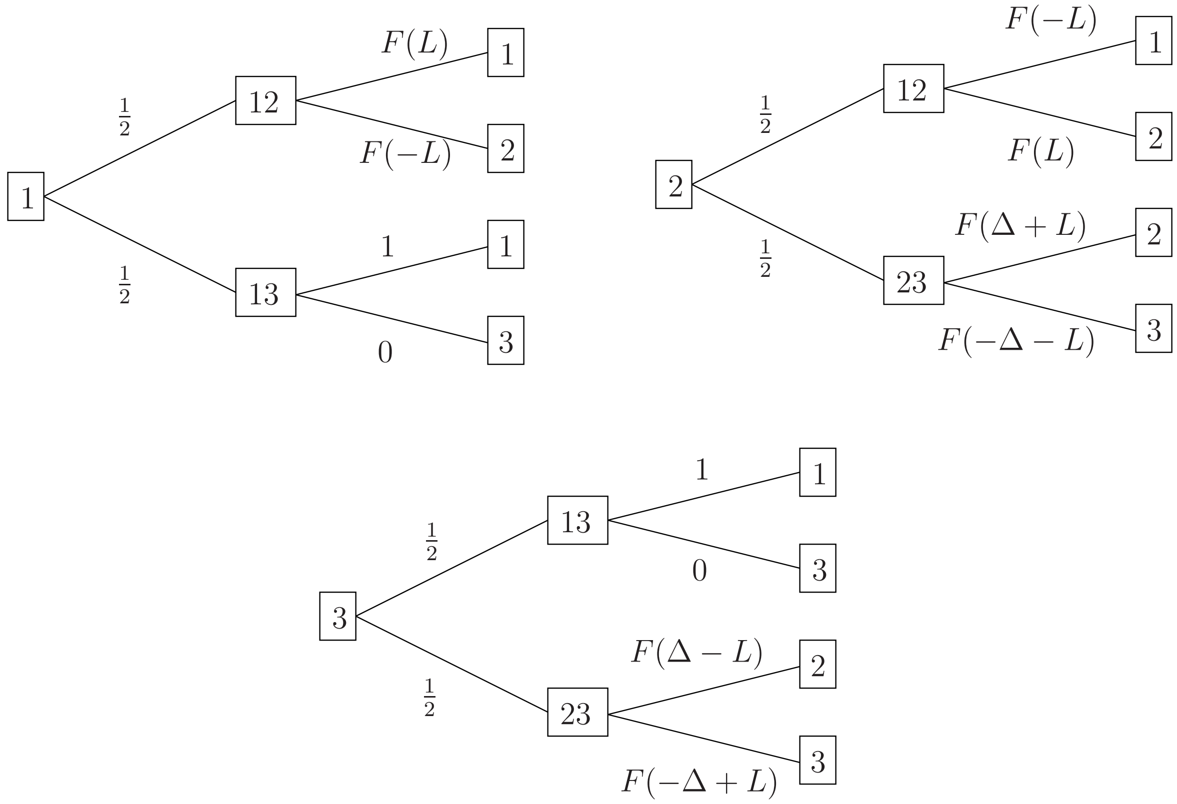


Figure 15: Level 1 decision tree for transition probabilities with 3 products including a loyalty effect.

where $A = 1/P$ is the affordability, s_V and s_A are the sensitivities to value and price respectively, and F is some function that maps $(-\infty, \infty)$ to $[0, 1]$ (for example $F(x) = (1 + \tanh(x))/2$). We can now model the effect of loyalty as an increase or decrease in Δ, as in the Logit model. Thus the probability of switching from j to i in a binary comparison is

$$F(\Delta_{ij} - L),$$

where L measures the strength of the loyalty effect, while the probability of staying with j is

$$F(\Delta_{ji} + L) = F(-\Delta_{ij} + L).$$

We consider again our three-product scenario, with products 1 and 2 equivalent to the consumer, so that $\Delta_{12} = 0$, and product 3 the inferior decoy, so that $\Delta = \Delta_{13} = \Delta_{23} > 0$. The decision tree is shown in Fig. 15.

The transition probabilities are given by

$$T = \begin{pmatrix} \dfrac{F(L)+1}{2} & \dfrac{F(-L)}{2} & \dfrac{1}{2} \\ \dfrac{F(-L)}{2} & \dfrac{F(L)+F(\Delta+L)}{2} & \dfrac{F(\Delta-L)}{2} \\ 0 & \dfrac{F(-\Delta-L)}{2} & \dfrac{F(-\Delta+L)}{2} \end{pmatrix}.$$

The steady-state market shares are given by

$$\begin{aligned} f_1 &= \frac{F(-\Delta-L)+F(-L)+F(\Delta-L)F(-L)}{2F(-L)+F(-\Delta-L)+F(-L)F(-\Delta-L)+2F(-L)F(\Delta-L)}, \\ f_2 &= \frac{(1+F(\Delta-L))F(-L)}{2F(-L)+F(-\Delta-L)+F(-L)F(-\Delta-L)+2F(-L)F(\Delta-L)}, \\ f_3 &= \frac{F(-\Delta-L)F(-L)}{2F(-L)+F(-\Delta-L)+F(-L)F(-\Delta-L)+2F(-L)F(\Delta-L)}. \end{aligned}$$

Let us compare this result to that which would arise from the Logit model. There the transition probabilities are given by

$$T_L = \begin{pmatrix} \dfrac{F(L)}{1+F(-\Delta-L)} & \dfrac{F(-L)}{1+F(-\Delta-L)} & \dfrac{F(\Delta-L)}{1+F(\Delta-L)} \\ \dfrac{F(-L)}{1+F(-\Delta-L)} & \dfrac{F(L)}{1+F(-\Delta-L)} & \dfrac{F(\Delta-L)}{1+F(\Delta-L)} \\ \dfrac{F(-\Delta-L)}{1+F(-\Delta-L)} & \dfrac{F(-\Delta-L)}{1+F(-\Delta-L)} & \dfrac{F(-\Delta+L)}{1+F(\Delta-L)} \end{pmatrix}.$$

The steady state market shares are given by

$$\begin{aligned} f_{L1} &= \frac{F(\Delta-L)(1+F(-\Delta-L))}{2F(\Delta-L)(1+F(-\Delta-L))+F(-\Delta-L)(1+F(\Delta-L))}, \\ f_{L2} &= \frac{F(\Delta-L)(1+F(-\Delta-L))}{2F(\Delta-L)(1+F(-\Delta-L))+F(-\Delta-L)(1+F(\Delta-L))}, \\ f_{L3} &= \frac{F(-\Delta-L)(1+F(\Delta-L))}{2F(\Delta-L)(1+F(-\Delta-L))+F(-\Delta-L)(1+F(\Delta-L))}. \end{aligned}$$

Notice that in the Logit model the market shares of products 1 and 2 are equal and less than 1/2. Thus there is no decoy effect in the Logit model.

The market shares of each product for the binary comparison model and the Logit model are illustrated in Figs. 16 - 21. Note that the Logit model predicts a much larger market share of product 3 (the decoy product), and that there is no decoy effect, that is, the market share of product 1 is the same as the market share of product 2, and is less than 1/2. The binary comparison model does show a decoy effect. The market share of product 1 is greater than 1/2, and when L is large it retains a market share significantly higher than 1/2, even though the market share of the decoy product is close to zero.

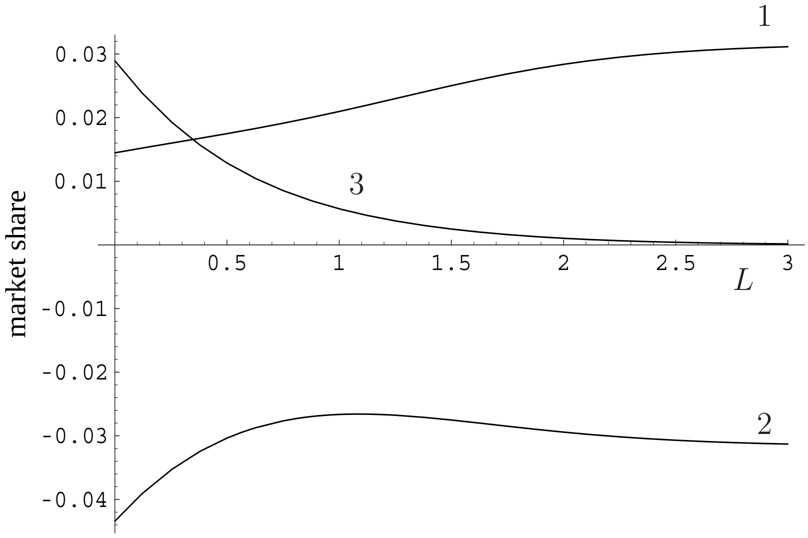


Figure 16: Individual market shares for the binary decision model as a function of loyalty L, with $\Delta = 1$. The market shares for products 1 and 2 are relative to 1/2, that is, the figure shown is the increase in market share after product 3 is introduced.

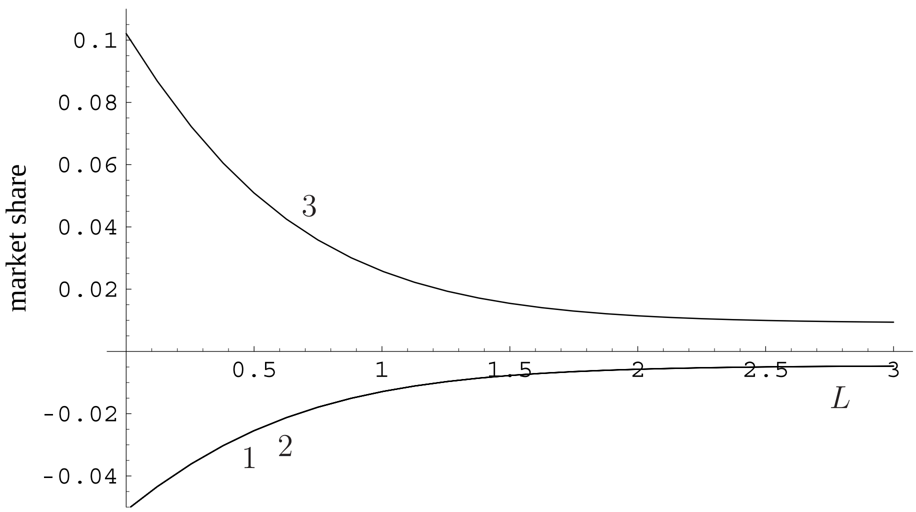


Figure 17: Individual market shares for the Logit model as a function of loyalty L, with $\Delta = 1$. The market shares for products 1 and 2 are relative to 1/2, that is, the figure shown is the increase in market share after product 3 is introduced.

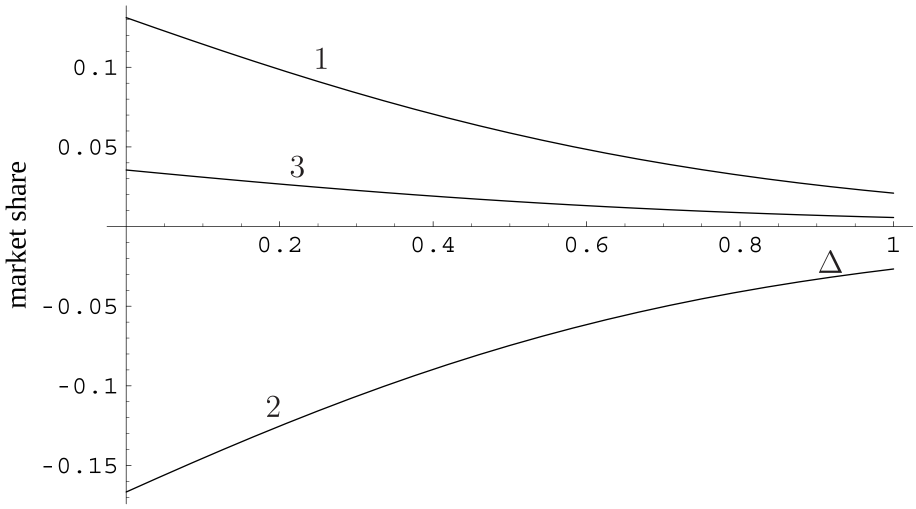


Figure 18: Individual market shares for the binary decision model as a function of the value of the decoy product Δ, with $L = 3$. The market shares for products 1 and 2 are relative to 1/2, that is, the figure shown is the increase in market share after product 3 is introduced.

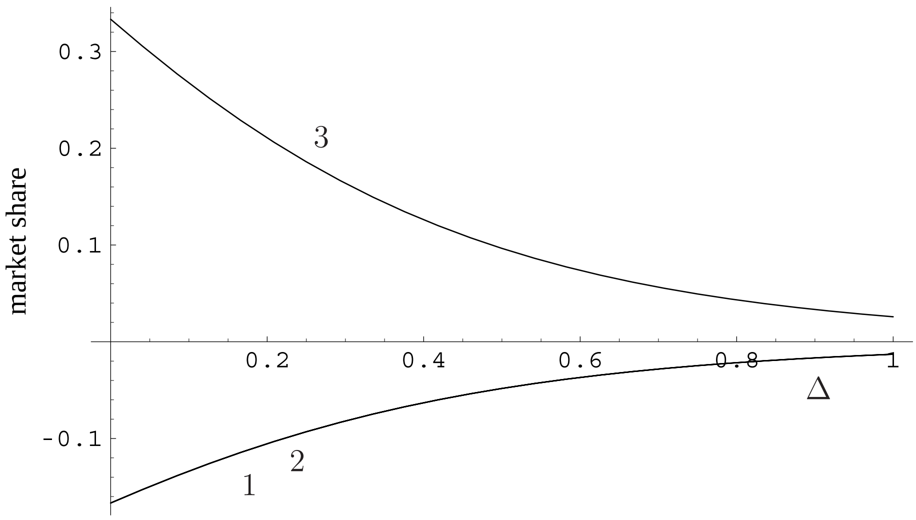


Figure 19: Individual market shares for the Logit model as a function of the value of the decoy product Δ, with $L = 3$. The market shares for products 1 and 2 are relative to 1/2, that is, the figure shown is the increase in market share after product 3 is introduced.

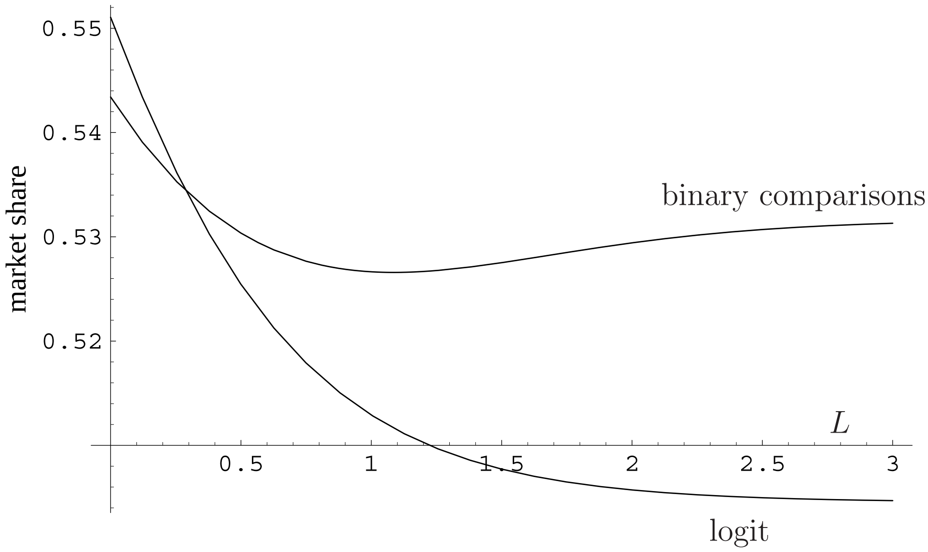


Figure 20: Total market share of products 1 and 3 combined as a function of the value of the decoy product Δ, with $L = 3$.

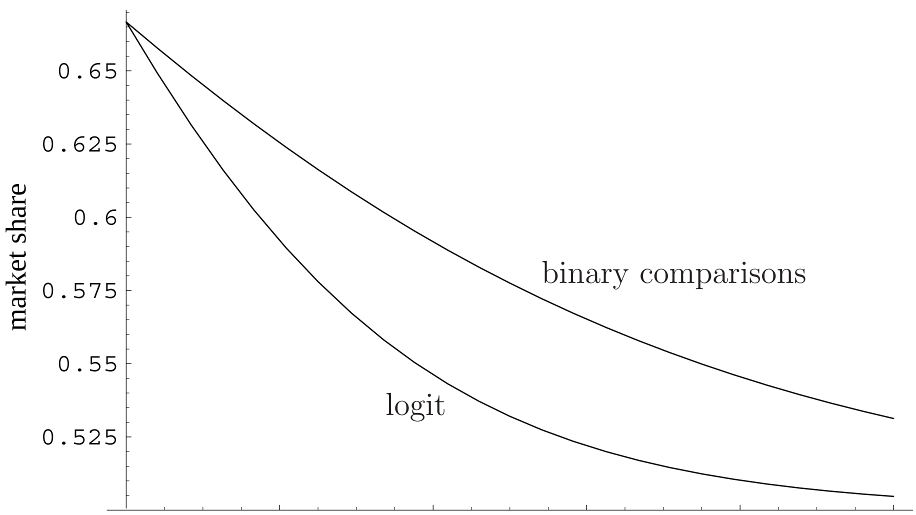


Figure 21: Total market share of products 1 and 3 combined as a function of loyalty L, with $\Delta = 1$.

One interesting observation from Fig. 20 is that the total market share of products 1 and 3 is not monotonic in the loyalty parameter L. The market share of product 3 decreases with loyalty, while the market share of product 1 increases with loyalty. The combined market share decreases with loyalty initially, but then increases.

When the decoy product is really bad, $\Delta \to \infty$, the market shares are given by

$$f_1 = \frac{1}{2}, \qquad f_2 = \frac{1}{2}, \qquad f_3 = 0,$$

in both the binary decision model and the Logit model. When the loyalty effect is strong, $L \to \infty$, then the limit is more subtle, and depends on the behaviour of the function F for large negative argument. The market shares are given by

$$\begin{aligned} f_1 &\sim \frac{F(-\Delta - L) + F(-L)}{2F(-L) + F(-\Delta - L)}, \\ f_2 &\sim \frac{F(-L)}{2F(-L) + F(-\Delta - L)}, \\ f_3 &\sim \frac{F(-\Delta - L)F(-L)}{2F(-L) + F(-\Delta - L)}. \end{aligned}$$

With $F(x) = (1 + \tanh x)/2$ we have

$$F(x) \sim e^{2x} \qquad \text{as } x \to -\infty,$$

this gives

$$f_1 \sim \frac{1 + e^{-2\Delta}}{2 + e^{-2\Delta}}, \qquad f_2 \sim \frac{1}{2 + e^{-2\Delta}}, \qquad f_3 \sim \frac{e^{-2L-2\Delta}}{2 + e^{-2\Delta}}.$$

Note that in the limit of strong loyalty, no one buys the decoy product, but there is a decoy effect as the market share of product 1 is increased. The corresponding limit in the Logit model gives

$$f_{L1} \sim \frac{1}{2 + e^{-4\Delta}}, \qquad f_{L2} \sim \frac{1}{2 + e^{-4\Delta}}, \qquad f_{L3} \sim \frac{e^{-4\Delta}}{2 + e^{-4\Delta}}.$$

In this case consumers continue to buy product 3, but the market share of products 1 and 2 is much closer to 1/2, since the correction is $O(e^{-4\Delta})$ rather than $O(e^{-2\Delta})$.

7.3 Networks

So far we have been considering each consumer in isolation. We can now generalise the model to a network of consumers. We will represent each consumer as a node in a graph, and link each consumer to some of the other consumers, which we will term his neighbours. Note that this label has nothing to do with physical proximity, but simply means that the two consumers are linked in the network. The links represent the other consumers who have an influence on this consumer; they may be friends or colleagues, or even celebrities used to advertise products. We can even imagine some nodes in the networks as being advertisements.

	product 1	product 2	product 3
value	0.75	0.25	0.2
affordability	0.25	0.75	0.7

Table 2: Parameter values used in the simulations

Now we have to decide how the neighbours influence our consumer. As with loyalty, there are two possibilities. The neighbours could influence the consumer's perception of a product, or they could simply bring the product to his attention, thereby influencing the probability that he chooses the product to compare in one of his binary comparisons. The simplest model is to assume that the first binary comparison a consumer makes is between his present product and one chosen randomly from one of his neighbours. We consider a slightly more general model, in which the consumer compares his present product with a randomly chosen neighbour's product with a certain probability P_{nbr}, but with probability $1 - P_{\text{nbr}}$ he chooses another product to compare with at random from the entire range of products.

7.3.1 Types of network

We simulated three different types of network. Each had 100 vertices (or nodes) and 200 edges. The **regular** network is simple a square lattice 'wrapped around' at the edges, so that each consumer is connected to his 4 nearest neighbours. The **random** network places 100 vertices down and then chooses 2 vertices at random to connect with an edge until all 200 edges have been placed. The **scale free** network (also know as a **small world** network) is generated by starting with the complete graph on five vertices (that is, each vertex is connected to every other vertex) and then adding vertices one at a time. Each vertex that is added is connected randomly to two other vertices, with the probability of connecting to a given vertex proportional to the degree of that vertex (that is, the number of edges currently connecting it to other vertices).

We also allow the possibility of including the effects of advertisments. These we model in the following way. We add one extra vertex for each product, and connect it to 10 other vertices chosen at random. The advertising node never changes its product.

In the simulations we have three products as usual. We choose the values and affordabilities of the products as shown in Table 2. We take the sensitivities to price and value to be equal to unity for each consumer (so that $s_V = s_A = 1$), and we take $F(x) = (1 + \tanh(x))/2$. This makes products 1 and 2 equivalent and has $\Delta = 0.1$. We take the loyalty parameter to be $L = 2.7$. Thus for isolated consumers who are not networked the market shares should be

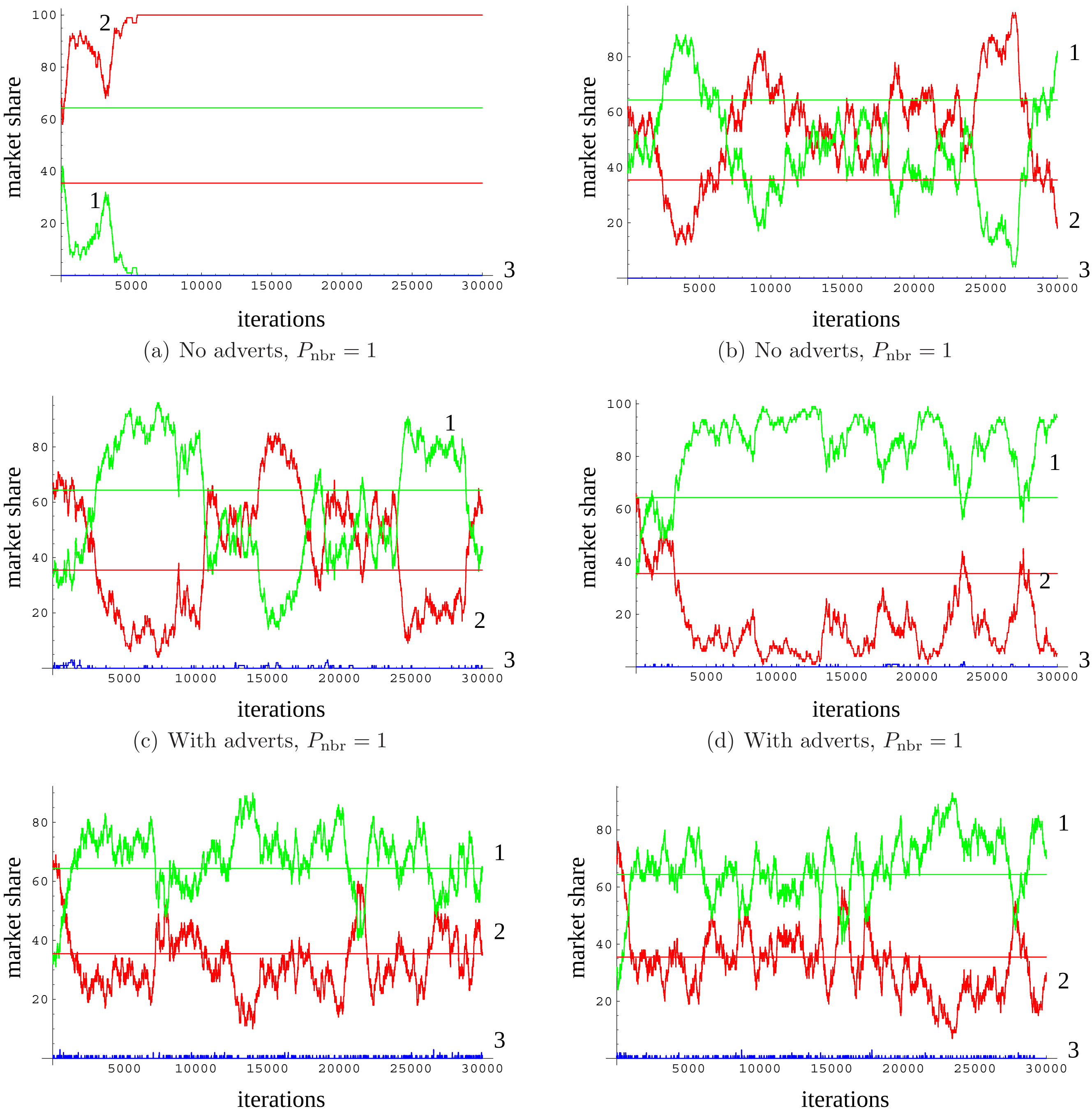
$$f_1 = 0.6439, \qquad f_2 = 0.3548, \qquad f_3 = 0.0013. \tag{41}$$

7.3.2 Results

Figs. 22, 23 and 24 show the results for the scale free, random and regular graphs respectively. We see that qualitatively there is very little to choose between the different types of network for the simulations presented.

- We see that in the absence of adverts and with $P_{\text{nbr}} = 1$ (so that the consumer always compares his current product to a chosen neighbour's product) we get lock-in, whereby one product reaches a 100% market share (Fig. 22(a)). Although lock-in is almost certain to occur, it may take a long time to do so. In Fig. 22(b) lock-in has still not occured, even after 30,000 iterations.
- In Figs. 22(c) and 22(d) we see the effect of including advertisements. The main effect is to prevent lock-in, since there is always at least one vertex associated with each product (the advertisements), from which market share can develop. Even with advertisements though, market share is still very volatile; Figs. 22(c) and 22(d) correspond to two simulations using exactly the same initial conditions and parameter values.
- In Figs. 22(e) and 22(f) we see the effect of increasing the probability of choosing a random product to compare with from zero to 0.3, both with and without adverts. This also has the effect of removing lock-in. Market share is also much more stable in this case, oscillating about the predicted average values. Note also that market share is more stable with adverts than without.

The simulations shown in Figs. 22 - 24 represent only the start of an investigation into the effects of different types of network and different parameter regimes. Clearly there is a lot of scope for future more detailed investigations in this area.



(a) No adverts, $P_{\text{nbr}} = 1$

(b) No adverts, $P_{\text{nbr}} = 1$

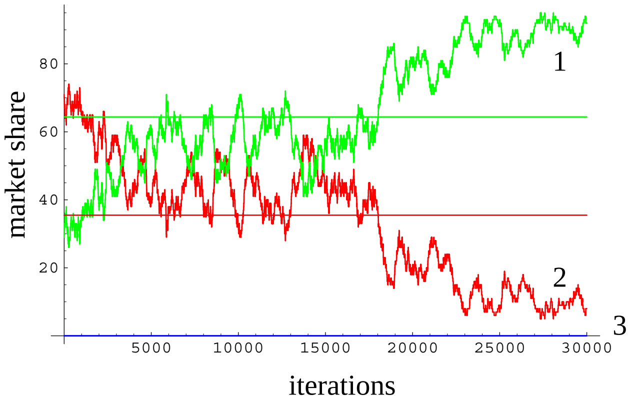
(c) With adverts, $P_{\text{nbr}} = 1$

(d) With adverts, $P_{\text{nbr}} = 1$

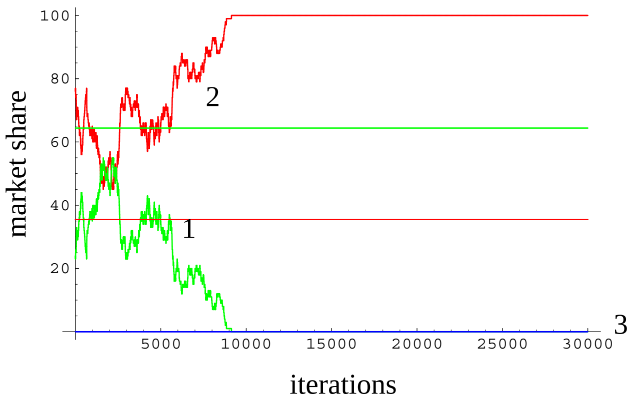
(e) With adverts, $P_{\text{nbr}} = 0.7$

(f) No adverts, $P_{\text{nbr}} = 0.7$

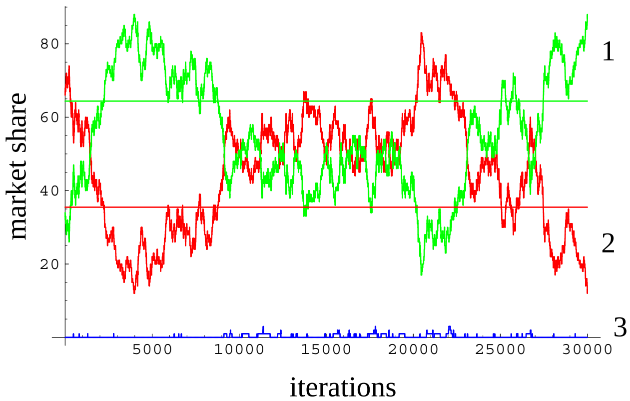
Figure 22: Scale free graphs



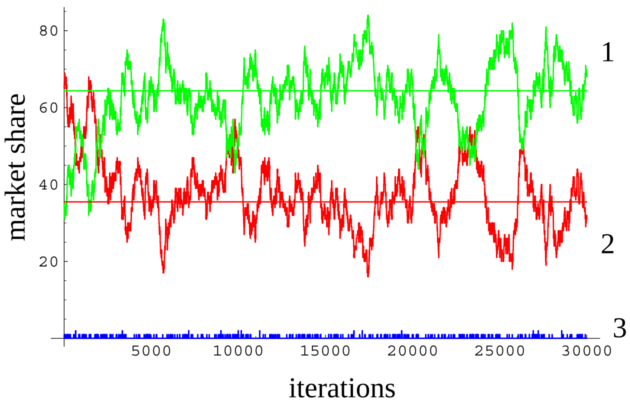
(a) No adverts, $P_{\text{nbr}} = 1$



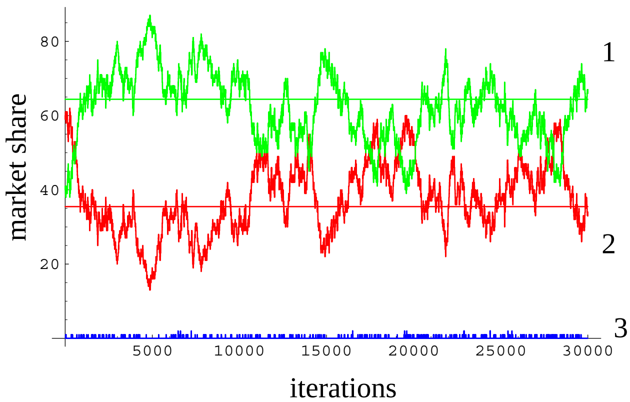
(b) No adverts, $P_{\text{nbr}} = 1$



(c) With adverts, $P_{\text{nbr}} = 1$

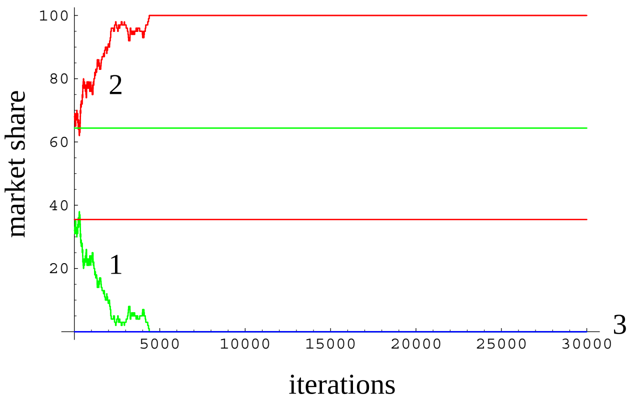


(d) With adverts, $P_{\text{nbr}} = 0.7$

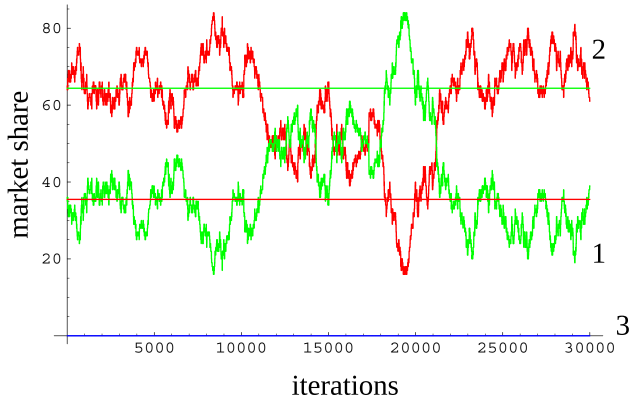


(e) No adverts, $P_{\text{nbr}} = 0.7$

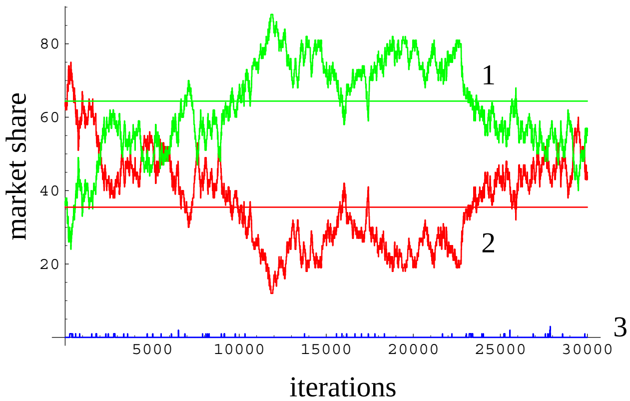
Figure 23: Random graphs



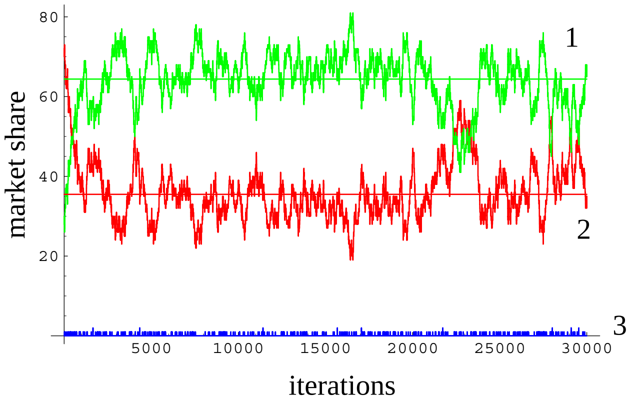
(a) No adverts, $P_{\text{nbr}} = 1$



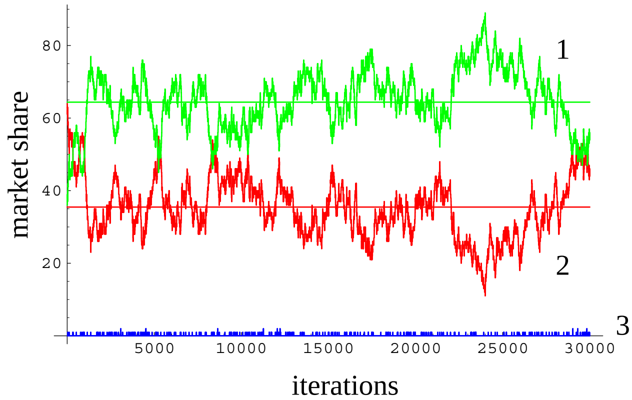
(b) No adverts, $P_{\text{nbr}} = 1$



(c) With adverts, $P_{\text{nbr}} = 1$



(d) With adverts, $P_{\text{nbr}} = 0.7$



(e) No adverts, $P_{\text{nbr}} = 0.7$

Figure 24: Regular graphs

8 Networks

In this section we consider the propagation of consumer behaviour across a square lattice (generalizations to other geometries are straightforward) in which each node represents an individual. Each node/individual, denoted by the pair (I, J), makes a choice between a number of products depending on the products' properties and on the views or opinions the 'preceding' nodes, *i.e.* those that lie below or to the left on the square lattice. We consider two or three products, characterised by their qualities, such as affordability. As an individual, a node has a psychology and a sociology, which determine its perception of each product. These can be modelled with various degrees of sophistication.

8.1 Propagation of a consumer preference through a network

We consider here a simple linear model. Each node (I, J) is given, as psychology, a sensibility κ^k_{IJ} to each quality k. Moreover, each node, as sociology, is influenced by the preceding nodes through a coefficient λ_{IJ}. We have taken the interaction between the nodes so that (I, J) is affected only by its two immediate neighbours, $(I-1, J)$ and $(I, J-1)$, and both exert the same influence per individual (the area of influence is easy to generalize). In our simulations, the coefficients κ^k_{IJ} and λ_{IJ} are positive and randomly chosen from a uniform distribution. Consumer (I, J) forms an opinion or view, in the form of a composite numerical value, on product i according to

$$V^i_{IJ} = \sum_k \kappa^k_{IJ} Q_{ki} + \lambda_{IJ}(V^i_{I-1,J} + V^i_{I,J-1}) \quad \text{for } I > 0\,,\ J > 0\,,$$

with Q_{ki} being the value of quality k for product i; V_{0J} and V_{I0} are taken as zero for I and J positive. Consumer (I, J) chooses to buy the product i for which V^i_{IJ} is largest; if there is no clear-cut 'best' buy, no product is bought. The process here starts from node $(1, 1)$ (but could start at any node or nodes of the lattice). The initial value V_{00} is always taken to be 1 here (although away from the bottom left in Figs. 25 and 26, the value of V_{00} has only minor effect.) This initial information can be regarded as an advertising process injected into the system. (Alternatively, such values can be used in a version with successive propagations over the lattice, to boost or handicap the coming of a new product onto the market, taking into account the loyalty of the consumers; such a model would be able to track the variation of market share over time.)

Six simulations have been carried out using the same uniform random distribution for $\mu_{IJ} = \kappa^1_{IJ}$, $\kappa_{IJ} = \kappa^2_{IJ}$ and λ_{IJ}. Fig. 25 shows the share of the market on a 10,000-node lattice for two products A and B, depending on different values for their qualities $Q_1 = P$ and $Q_2 = Q$. We observe its spatial evolution according to the position of the product A on, above or below the trade-off line through B (fixed by having the same distributions for μ and κ). Fig. 26 shows the effect of the introduction of a third product C on the

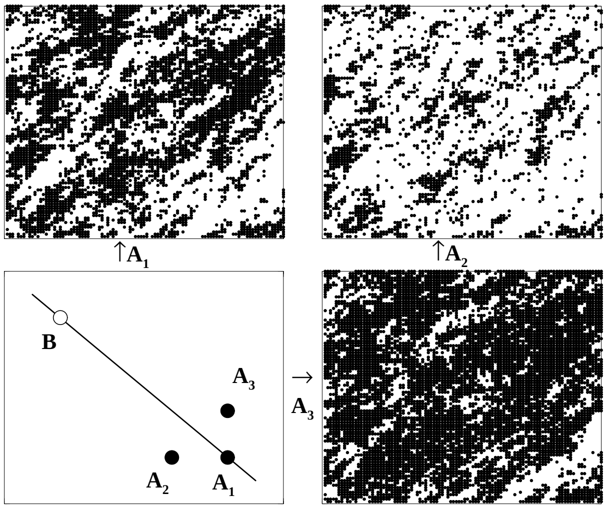


Figure 25: Share of the market between A (black) and B (white) depending on the position of A with respect to the trade-off line. The propagation over the lattice starts from the lower left corner. If A lies above the trade-off line, it achieves much higher market share than if below it.

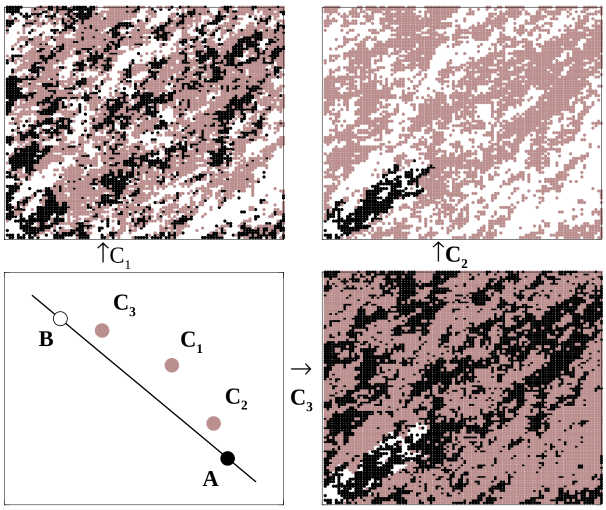


Figure 26: Share of the market between A (black) and B (white) and C (brown) depending on the position of C, always taken to be above the trade-off line. The propagation over the lattice starts from the lower left corner. Product C acquires market share more heavily from the product to which it is closer in product space.

market, and the transfer of buying behaviour from A or B to C depending on the position in ‘quality space’ of C in relation to the other two products.

The introduction of the third product C with low P and Q, so that it lies below the trade-off line, cannot produce a decoy effect in the basic form of the model, since consumers purchase simply according to highest V.[9] A move of the market towards C in a linear model as presented here requires the possibility of negative values for α_{IJ} or κ_{IJ}. The introduction of negative values for μ or κ decreases the rôle of the psychology (*i.e.* the contribution of terms $\mu_{IJ}P + \kappa_{IJ}Q$) and makes the sociology the main factor in the propagation, through the term $\lambda_{IJ}(V_{I-1,J} + V_{I,J+1})$. Negative values of μ or κ are consistent with irrational behaviour, with people preferring over-priced or inferior products. Negative values of λ could signify people wanting to be different from their neighbours. The model can be easily modified, by introducing functions more sophisticated than simple linear ones, *e.g.*

$$V_{IJ} = \mathcal{A}_{IJ}(P) + \mathcal{K}_{IJ}(Q) + \mathcal{L}_{IJ}(V_{I-1,J}, V_{i,j+1})\,.$$

[9]Note, however, that the introduction of a third product may itself change the trade-off line (see Fig. 2).

9 Some other models

This section is rather more speculative and considers the possibility of people not so much preferring a product as aiming to make their purchases to suit their quality requirements. Consumers vary their position in quality space over time, and it is this position which determines which product(s), if any, they buy. This concept is extended to allow for a large number of products, the number being large enough for the products to be treated as a continuum with a density – a new continuous variable – in quality space.

9.1 A particle-dynamics model

The basic idea is to build a model based on a physical analogy, between the consumers' buying behaviour and particle-particle dynamics. We assume consumers to be particles moving in quality space. The products are considered as sources of attractive potentials, the details of the potentials depending on the products' characteristics. The psychology of people will be regarded as the influence of the potential on the 'mass' of the particles. This will depend on the consumer but also on the characteristics of each product, weighted by coefficients of 'choice'. The sociology of individuals can be considered as a form of particle-particle interaction (although this is not pursued explicitly here). We then look at the evolution of consumer-particles. We define a critical radius around each product and assume that people buy the product if they get within this distance of it.

Definition of the product space and potential. Following what has gone before, the product space is defined by just two characteristics for each product i: affordability P_i and quality Q_i. We can define an 'attractiveness' of a product as the 'distance' of the product from the origin in product space: $U_i = (P_i + Q_i)/2$. With this definition, all the products with $P_i + Q_i =$ constant will have the same attractiveness. The attractive potential of the product i will be proportional to U_i. In the following we have taken the simple potential

$$F_x(x) = -\sum_i (x - P_i)U_i \quad , \; F_y(y) = -\sum_i (y - Q_i)U_i \, ,$$

where x and y are the coordinates of a consumer in (P, Q) space.

Implementation of psychology. It is the psychological 'mass' ψ of a consumer that makes the difference in the choice of two products of the same attractiveness (although to call ψ a mass is somewhat inaccurate, since it also depends on the consumer's environment). To put some detail on the model, introduce the variations from the mean of P and Q of each product: $D_{pi} = P_i - U_i$ and $D_{qi} = Q_i - U_i$. For each product, ψ will have the following three characteristics:

- For each product it is inversely proportional to its affordability (x direction) or quality (y direction).

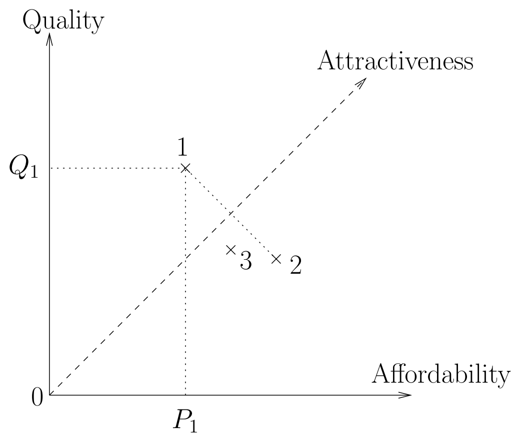


Figure 27: Product space with three products.

- It also depends inversely on the distances D_p and D_q, which quantify the importance of P and Q. The distance is modified by comparing it to the sum of all remaining distances:
$$D'_{pi} = \frac{D_{pi}}{\sum_{j \neq i} D_{pj}}$$
and similarly for D'_{qi}. The modified distances indicate the relative importance of particular P_i's (and Q_i's) in comparison to the others. This comparison will also play a key rôle when introducing a decoy product.

- Both these dependencies are weighted by numerical coefficients (α and β) that give the importances of affordability and quality for the consumer: the more important the criteria is to the consumer, the lower the 'inertia' that is attributed to the 'best' product regarding this criteria.

Putting these observations together, we have, for consumer i,

$$\psi_{xi} = \left[\left(\frac{\alpha_i}{D'_{p1}} + \frac{\beta_i}{D'_{q1}} \right) / P_1 + \cdots + \left(\frac{\alpha_i}{D'_{pn_p}} + \frac{\beta_i}{D'_{qn_p}} \right) / P_{n_p} \right]$$

and the equivalent for ψ_{yi} with Q replacing P. Here n_p is the number of products.

Implementation of sociology. Sociology could be seen as some particle-particle interaction but we have not tried to implement it here.

Equations of motion. We assume that the 'velocity' of consumer i in product space is proportional to the attractiveness divided by the mass. For each consumer we have:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_i F_x(x_i)/\psi_{xi} \,,$$

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = \sum_i F_y(y_i)/\psi_{yi}\,.$$

Preliminary results

- In Fig. 28 we have two products of the same attractiveness and three different consumers. In green and black we have two consumers who are quite sensitive to affordability and quality, respectively, whereas the blue one slightly prefers quality. Their choices reflect their psychology.

- In the Fig. 29 we have the same three consumers with three products of the same attractiveness. This time the behaviour of the first two consumers does not change but the last one will choose the product that offers the best balance between affordability and quality. Here again the results reflect the psychology of the consumers.

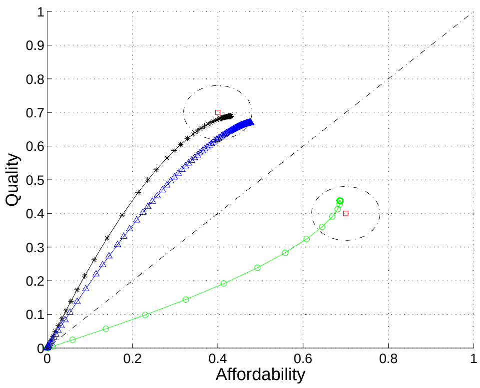


Figure 28: Evolution of three different consumers towards two products of the same attractiveness

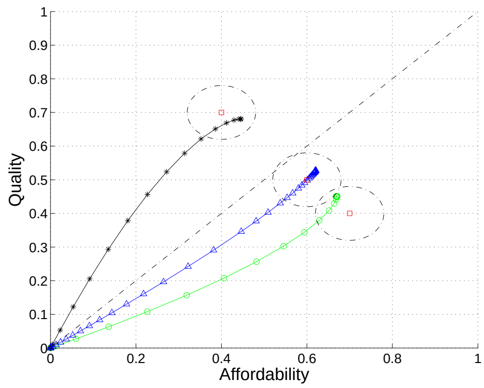


Figure 29: Introduction of a third, 'middle-range' product of the same attractiveness

- Fig. 30 introduces a decoy product. We start with the choice of products reached in the first simulation (Fig. 28). The decoy will influence the psychology of consumers by fundamentally modifying $\sum D_{pj}$ and $\sum D_{qj}$. The quality and affordability of products relative to the others, *i.e.* the perception of the product by the consumer, is then modified. This results in the change in the behaviour of the third consumer (cyan line). An improvement of the model would increase this effect for the third consumer.

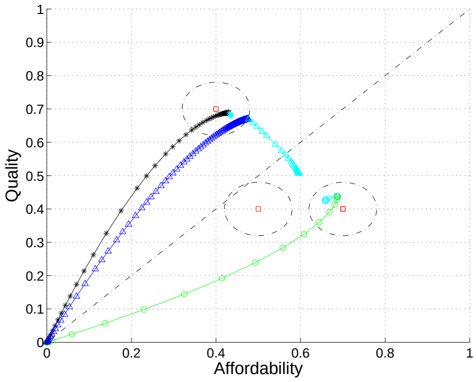


Figure 30: Influence of a decoy product.

Conclusions. This rough model is able to reproduce a few features of observed consumer behaviour, but there is scope for many things to be improved. For example, the choice of initial conditions and the choice of type of 'attractive potential'. Social (particle-particle) effects are also to be included.

9.2 A continuum of products

The Study Group gave some consideration to 'moving' products, *i.e.* products whose price and/or other attribute(s) change over time. There was also some discussion of having so many products that, instead of a finite number evolving over time in quality space, they would form a continuum, with an evolving density in quality space. More progress was made with a static continuum: with respect to quality coordinates $\boldsymbol{Q}$, the products have a 'density' $Y(\boldsymbol{Q})$ (= number of different products within a 'unit volume' of quality space). The market shares of individual products, X_i, are likewise replaced by a rate of consumption per unit volume, and so also a density, but one which varies over time, $X(\boldsymbol{Q}, t)$. This population density can be thought of as being made up of individual consumers, who change their preferences, gradually, over time, as they keep purchasing at the same uniform rate.[10] There is then a velocity, $\mathrm{d}\boldsymbol{x}/\mathrm{d}t$, associated with consumers at the point $\boldsymbol{x}$ in quality space at any particular time. This velocity will be determined by psychological, and possibly sociological, considerations. The trajectories can be identified with a resulting hyperbolic partial differential equation for the population density X.

The concept of outlier avoidance suggests that consumers should prefer products similar to others, *i.e.* they change their choice of product to move towards zones (in quality space) of higher product density. This suggests a velocity (see Fig. 31) something like

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \nabla Y \,.$$

[10]The total market share is $\int_{\mathcal{P}^+} X \, \mathrm{d}Q \equiv 1$, integrating over the set $\mathcal{P}^+ = \{\boldsymbol{Q} : Q_k > 0 \text{ for all } k\}$ of realistic quality values.

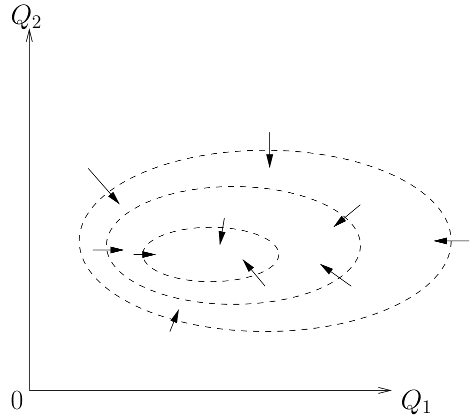


Figure 31: Consumer movement due to outlier avoidance. The dashed curves are lines of constant product density.

On the other hand, with a utility function, $U(\boldsymbol{Q};Y)$, depending on the distribution Y of products, consumers can be expected to move towards 'better' products and this would suggest (see Fig. 32),

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \nabla U\,.$$

The idea of minimum anticipated regret suggests similar behaviour. (Decision process

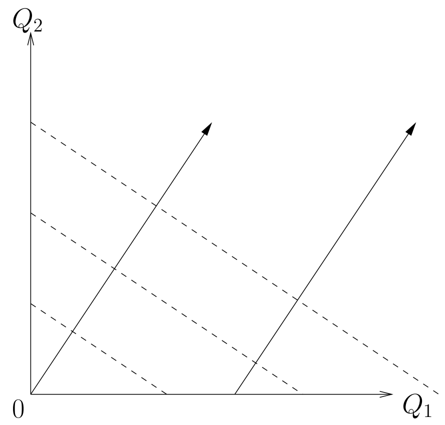


Figure 32: Consumer movement due influenced by 'overall quality'. The dashed lines are where the utility function is constant.

change will not arise here as we have a very large number of products.)

Combining these psychological considerations, $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = a_1\nabla Y + a_2\nabla U$. Here U will be an increasing function of the qualities Q_k (components of $\boldsymbol{Q}$), such as $\boldsymbol{\beta}\cdot\boldsymbol{Q}$ with $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{n_q})$ and all the β_k's non-negative, whereas Y will have a global maximum at some point $\boldsymbol{Q}^*$ (possibly on the boundary of $\mathcal{P}^+$, *i.e.* $Q_k^* = 0$ for some k) and should decay rapidly as $|\boldsymbol{Q}| \to \infty$. To avoid consumers drifting off to unrealistically high qualities, the velocity might be modified to give something like

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = a_1 \nabla Y + a_2 Y \nabla U \,. \tag{42}$$

The velocity $\mathrm{d}\boldsymbol{x}/\mathrm{d}t$ leads to a flux of consumers, $\boldsymbol{q} = X\mathrm{d}\boldsymbol{x}/\mathrm{d}t$, and then conservation of custom (market share), $\partial X/\partial t + \nabla \cdot \boldsymbol{q} = 0$, means that

$$\frac{\partial X}{\partial t} + \nabla \cdot \left((a_1 \nabla Y + a_2 Y \nabla U)\, X \right) = 0 \,. \tag{43}$$

If new consumers are not to appear out of thin air, there should be no flow of them across the boundary of $\mathcal{P}^+$:

$$\left(a_1 \frac{\partial Y}{\partial Q_k} + Y \frac{\partial U}{\partial Q_k} \right) X = 0 \qquad \text{on } Q_k = 0 \,.$$

Solving (43) by the method of characteristics, *i.e.* integrating along the path given by (42), would indicate that market share gets concentrated around points where $a_1 \nabla Y + a_2 Y \nabla U = \boldsymbol{0}$. In particular, if we could write $a_1 \nabla Y + a_2 Y \nabla U = \nabla V$ for some function $V(\boldsymbol{Q})$ (which will not, in general, be the case), the market share would build up near maxima of V and become zero elsewhere.

'Positive' social effects, with people tending to purchase similar products as their friends (without proper networking considerations) might lead to migration towards more popular items, suggesting a velocity of the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \nabla X \,, \text{ or } |\nabla X|^\gamma \nabla X \,.$$

Combining this with (42) now leads to

$$\frac{\partial X}{\partial t} + \nabla \cdot \left((a_1 \nabla Y + a_2 Y \nabla U)\, X \right) + a_3 \nabla \cdot (X \nabla X) = 0 \,, \tag{44}$$

which is a backward heat equation. Such problems are well known to be ill posed. This 'wrong' diffusivity might possibly be corrected by including an additional random drift term (changing preferences as a Brownian motion) to get

$$\frac{\partial X}{\partial t} + \nabla \cdot \left((a_1 \nabla Y + a_2 Y \nabla U)\, X \right) = \nabla \cdot ((1 - a_3 X) \nabla X) \,.$$

However, a 'negative' social effect, people wanting something different from their friends, reverses the direction for the original velocity and a forward (well-posed) diffusion equation results:

$$\frac{\partial X}{\partial t} + \nabla \cdot \left((a_1 \nabla Y + a_2 Y \nabla U)\, X \right) = a_3 \nabla \cdot (X \nabla X) \,.$$

An attempt might also be made to include geography as another continuum variable; this means allowing for a continuum of agents. If $\boldsymbol{y}$ represents location of consumers, *e.g.* position along a street, the market-share density is now additionally with respect to, and dependent on, this new independent variable: $X = X(\boldsymbol{Q}, \boldsymbol{y}, t)$. The psychology gives rise to the same sort of terms as before. Sociology can now lead to influences of neighbours as well as, or instead of, friends, flatmates or family members. If this is such that distributions should tend to equilibriate, we might expect to include a term of the form $D\nabla_y^2 X$ on the right-hand side of (44). Alternatively, if people at $\boldsymbol{y}$ are influenced according to what neighbours do overall, the model might be changed by adding $a_4 \nabla_y^2 \int_{\mathcal{P}+} \boldsymbol{Q} X \, \mathrm{d}Q$ to the velocity.

Modelling agent position as a relevant continuum variable is easier if we return to just two distinct products, so that $X(y, t)$ is the market share of the first at location y and time t. If, in contrast to Section 4, the rate of change of market share is given by subtracting the market share from the average of that of the neighbours (in a discrete analogy), we expect $\partial X/\partial t = D \partial^2 X/\partial y^2$. Combining this with the terms which appeared in Section 4, the market share now evolves as

$$\frac{\partial X}{\partial t} = D \frac{\partial^2 X}{\partial y^2} + (1 - 2X)(1 - KX + KX^2) \, .$$

Apart from the 'trivial' steady states, $X \equiv X_\pm$, this also has steady solutions which link these two lock-in values and others which are periodic in y. In two space dimensions, and taking D to be 'small', this equation exhibits 'motion by mean curvature', with the boundary between regions of $X \sim X_\pm$ gradually getting shorter. A biased version, say

$$\frac{\partial X}{\partial t} = D \frac{\partial^2 X}{\partial y^2} + 1 - \alpha X + KX(1 - X)(2X - 1) \, ,$$

with α between 1 and 2 if the first product is to be preferred, has travelling-wave solutions linking $X_\pm$. In common with the agent-based Ising-type model, on which Unilever has performed large-scale simulations, the model is of a sort that is used to model phase transitions. It is therefore unsurprising that the models exhibit lock-in behaviour, analogous to having distinct phases.

10 Concluding remarks

The discussions over the course of the week barely scraped the surface of a potentially huge subject. The hierarchy of models identified so far, which is by no means exhaustive, suggests that a vast amount of modelling could be done. It is clear that even with the models touched on during this study there is scope for much further work, for instance extending consideration of networks to get reasonable representations of social effects, carrying out more numerical simulations, especially with discrete-time, stochastic and agent-based models, or allowing for memory (or 'history'). Even relatively simple things remain to be done, for example looking at social interactions (particle-particle influences) in the particle model of Section 9 or looking at how a decoy might be used to switch lock-in according to models such as those of Section 4.

Of perhaps academic interest is the question of how realistic simple models might be, *e.g.* can a simple, lumped, deterministic model exhibit real-life behaviour (qualitatively and perhaps quantitatively) as well as an exhaustive suite of stochastic simulations with many agents? Whatever else, the use of any of the models suggested to date (lumped or agent, deterministic or stochastic, *etc.*) requires a good knowledge of the terms (*e.g.* the sort of nonlinear functions and preferably values of numerical constants) that appear in the equations. Acquiring such knowledge might well entail collaboration with and/or acquisition of data from market researchers and/or psychologists.

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