Electrode Breakage: Elkem

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Executive Summary

The following report details two main themes:

1. Classical Failure Criteria. Sections 1 and 2 are the response to the specific Elkem questions about the electrode′response to thermal stresses. In particular, three failure criteria based on classical macroscopic theories of crack initiation and propagation were compared. Unfortunately the usefulness of these predictions is obscured by uncertainty about the time scales over which various mechanical and thermal mechanisms act.

2. New Theories for Damage. A radically different theoretical approach (described in §3 and §4) involves modelling the mechanical failure of the electrode in terms of "damage", *i.e.* the local volume fraction ω of material that has undergone microfracture. Thus ω is zero in pristine material and saturates at a critical value, say unity. The key idea is to model the temporal growth of ω with a rate equation, the rate being an increasing function of the local stress σ but also depending upon ω. As in the case of simple chemical reactions, damage can be limited if σ is not too large, but it can increase to unity when σ exceeds a threshold value.

A simple paradigm was devised to ascertain the general features of the rate equation and this highlighted the importance of the global stress distribution throughout the electrode. The most important new idea was the incorporation of spatial variations into the model, which led to a novel coupled equation for σ and ω. These immediately revealed the possibility of confined regions of damage becoming unstable to small spatial inhomogeneities and hence spreading through the electrode as "travelling waves".

One paradox emerged concerning the dependence of the damage growth on the electrode size, which does not appear to correlate well with Elkem experimental data. It is hoped this paradox can be resolved and the whole theory tightened up at the forthcoming workshop at the Newton Institute in Cambridge.

1 Introduction

The general aim was to investigate what were the key effects and factors influencing, and in particular causing, a major breakage of a Søderberg electrode. During normal operation in an electric furnace an electrode is subject to high temperature gradients and temperature changes as the electrode is moved down (typically at about 1m/day) but with these slow changes over time, thermal stresses cause no significant damage. If the furnace is temporarily shut down for some reason, the differing changes in temperature in the outer and inner parts of the electrode cause varying thermal contraction and expansion during cooling and subsequent reheating, resulting in extra internal stresses. Fracture, an expensive event, can sometimes happen during this procedure.

One eventual target would be to understand how to select pastes, from which the electrodes are made, having mechanical and thermal properties which make breakages less likely during shut downs. (Some better understanding of how the electric currents, and resultant heating in electrodes, are best reduced and then increased again, in ways conducive to electrodes staying in one piece, might also be beneficial.)

The work done during the Study Group focused on three aspects:

(1) identifying the importance of various groupings of material properties regarding fracture to see what sorts of materials might be more, or less, liable to break;

(2) briefly considering a simple thermal problem to check which regions of an electrode cool faster than others;

(3) looking at a model for "damage" with the eventual aim of seeing how a substantial break in an electrode can form and grow.

The group spent most of its time working on the last of these but only a few very simple configurations were considered. Ideally, such models would be taken much further, and combined with considerations such as *(1)* or *(2)*, to see if a particular paste is likely to make a poor electrode or how furnaces should be shut down and restarted to minimise danger of breakage.

2 Important Thermo-mechanical Properties

The identification of which parameter groupings (Hasselman, Schneider & Coste) might be significant was done by considering the possible consequences of changing the internal heating rate. This very simple approach entails looking at changes in steady temperature resulting in a change in volumetric power; thus the results can only be applied over time scales greater than all thermal and mechanical relaxation times.

For a change of energy produced per unit volume per unit time ΔQ, the temperature (eventually) changes by an amount of size

$$\Delta T = \Delta Q\, L^2/\lambda\,,$$

where λ = thermal conductivity of the object being heated (or cooled) and L is a typical length scale of the object (*e.g.* the radius of the cross-section for an electrode). The resulting (relative) displacements are then typically

$$\alpha L\, \Delta T = \alpha\, \Delta Q\, L^3/\lambda\,,$$

with α = coefficient of thermal expansion, and thermal stresses are of order

$$E \times \text{displacement}/L = \alpha E\, \Delta Q\, L^2/\lambda$$

for E = modulus of elasticity.

One quantity relevant for fracture is the "tensile strength" of a material:

$$f_t = \text{stress at which a crack can form}$$

For a material to start to fracture (given this long-time temperature change) a thermal stress should exceed the tensile stress:

$$\alpha E\, \Delta Q\, L^2/\lambda > f_t\,.$$

Equivalently, cracks should not start to form in the material if

$$R \equiv \text{"resistance to crack initiation"} \equiv \lambda f_t/\alpha E > \Delta Q\, L^2.$$

(For the electrode pastes under consideration in the Study Group, $R \simeq 410\,\mathrm{W\,m^{-1}}$, $1100\,\mathrm{W\,m^{-1}}$, $450\,\mathrm{W\,m^{-1}}$ for materials A, B, C respectively; L is typically 1 m and $\Delta Q \simeq 10^5\,\mathrm{W\,m^{-3}}$ so, given enough time, fractures will form.)

Another material quantity is the "fracture energy", G_F,

$$G_F \times \text{Area} = \text{energy used to form a break.}$$

This quantity is measured by first making a cut in a bar of the material and then breaking the bar, with the area of the fracture being the part of the cross-section not already cut. (Force is applied so as to try to bend the bar.) The energy needed to form the break is given by the work done by the applied forces in obtaining specified displacements ($\int F dx$). (On carrying out experiments with different sized bars, energy appears to grow slightly more slowly than length squared, possibly due to some energy being expended in damaging the bar where the forces are applied.)

Returning to the thermally stressed material of typical length L, the elastic energy stored (force $\times$ displacement) is of order

$$L^2 \times \text{stress} \times \text{displacement} = L^7 (\Delta Q)^2 \alpha^2 E/\lambda^2.$$

The energy required for a significant facture is of size

$$G_F L^2$$

so a substantial break requires

$$L^5 (\Delta Q)^2 \alpha^2 E/\lambda^2 \geq O(G_F);$$

equivalently, no major fracture is possible if

$$R'_{st} \equiv \text{"index for crack propagation"} \equiv \frac{\lambda}{\alpha}\sqrt{\frac{G_F}{E}} \gg \Delta Q\, L^{5/2}.$$

(The three types of paste have $R'_{st} \simeq 260\,\mathrm{W\,m^{-1/2}}$, $630\,\mathrm{W\,m^{-1/2}}$, $120\,\mathrm{W\,m^{-1/2}}$ for A, B, C, compared with $\Delta Q\, L^{5/2} \simeq 10^5\,\mathrm{W\,m^{-1/2}}$, which would indicate that all have the potential, given time, to produce substantial breaks.)

Certainly it appears from this elementary discussion that both the resistance to crack initiation, R, and index for crack propagation, R'_{st}, were significant for the fracture by thermal shock, given sufficient time (with materials B best according to both criteria, material A and C similar with regard to the first but C definitely worst looking at the second criterion.) (See Hasselman and Schneider & Coste.) Some work such as Wang & Krstic indicates that small cracks, pores and crystals can affect the effective modulus of elasticity E and thereby help to increase R and R'_{st}. The paper by Allard *et al.* illustrates the rôle of larger grains hindering growth of a fracture through blocking its path and making it take longer paths. One other quantity $B \equiv$ "brittleness number" $\equiv f_t^2/EG_F = (R/R'_{st})^2$ did not appear to have such direct relevance for this thermo-mechanical problem.

3 Transient Thermal Problem

The above analysis only considers the possible consequences of stresses induced by temperature changes over a long time. Taking typical values for density, $\rho \simeq 1.4 \times 10^3\,\mathrm{kg\,m^{-3}}$, specific heat, $C = 2 \times 10^3\,\mathrm{J\,kg^{-1}\,K^{-1}}$, thermal conductivity, $\lambda \simeq 5\,\mathrm{W\,m^{-1}\,K^{-1}}$, and electrode radius, $L \simeq 1\,\mathrm{m}$, the thermal "diffusion time" (balancing the time and spatial derivatives in the last equation) is of size $\rho C L^2/\lambda \simeq 6 \times 10^5\,\mathrm{s} \simeq 6$ days. This indicates that for a shut-down time of about 6 hours the electrode will not cool significantly; to be more precise, differential cooling will be confined to a boundary layer near its surface. (This needs a little caution. For the extreme case of a sphere of radius L, with fixed surface temperature, the temperature decay is like $e^{-\pi^2\lambda t/\rho C L^2}\sin(\pi r/L)$. For material with these properties, the cooling time is then $\simeq 6 \times 10^4\,\mathrm{s} \simeq 16$ hours.) To be more precise, a "short-time" temperature, valid for times much less than 6 days after a steady volumetric heating rate of Q is switched off, might be looked for.

Writing temperature $T(\boldsymbol{x}, t) = T_0(\boldsymbol{x}) - \theta(\boldsymbol{x}, t)$ with $T_0 =$ steady temperature due to volumetric heating rate, Q, $\theta =$ cooling a time t after switch off $\sim -Q/\rho C$ away from the surface (distances from the surface large compared with $\sqrt{\lambda t/\rho C}$). Taking distance y from the surface comparable with $\sqrt{\lambda t/\rho C}$ ($y \ll L$) and assuming, for simplicity, that Newtonian cooling applies so that $\lambda \frac{\partial T}{\partial n} = h(T_A - T)$ for some ambient temperature T_A and heat-transfer coefficient h,

$$\theta \sim \frac{Q}{\rho C}\left(t - \left(h/\sqrt{\lambda\rho C}\right) t^{3/2}\varphi(\eta)\right),$$

where $\eta = \sqrt{\frac{\rho C}{\lambda t}}\, y$ and $\varphi(\eta)$ satisfies

$$\frac{d^2\varphi}{d\eta^2} + \frac{\eta}{2}\frac{d\varphi}{d\eta} - \frac{3}{2}\varphi = 0 \text{ for } \eta > 0, \qquad \frac{d\varphi}{d\eta} = -1 \text{ at } \eta = 0, \qquad \text{and } \varphi \to 0 \text{ as } \eta \to \infty.$$

After some manipulation,

$$\varphi(\eta) = \frac{1}{3\sqrt{\pi}}(\eta^2 + 4)e^{-\eta^2/4} - \frac{1}{6}\eta(\eta^2 + 6)\,\mathrm{erfc}\left(\frac{\eta}{2}\right).$$

($\mathrm{erfc}\, z = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-s^2} ds$.)

For these short times, the relevant length scale is $\sqrt{\lambda t/\rho C}$, not L, relative temperature changes ΔT are of size $Qh\lambda^{-1/2}(\rho C)^{-3/2}t^{3/2}$ so that thermal stresses and elastic energies are of order $\alpha E Q h \lambda^{-1/2}(\rho C)^{-3/2}t^{3/2}$ and $E\alpha^2 Q^2 h^2 \lambda^{1/2} = t^{9/2}(\rho C)^{-9/2}$ respectively (taking the thermal boundary-layer width to be the relevant overall distance). Now the two criteria for fractures forming and becoming significant (again over the thermal diffusion length) appear to be

$$\frac{\lambda^{1/2}(\rho C)^{3/2} f_t}{\alpha E h} < Q t^{3/2} \quad \text{and} \quad \frac{\lambda^{1/2}(\rho C)^{7/2} G_F}{\alpha^2 E h^2} \leq O(Q^2 t^{7/2}).$$

(This analysis can be adapted for radiative heat transfer.)

Now that the cooling (and reheating) time is limited, the theory of §2, concerning what fractures might form given an eternity, is not necessarily sufficient. A "non-classical" theory as to how breaks form, and in particular how they do so over time, must be considered.

4 Damage

To see how an electrode might actually break apart, a model for the accumulation of "damage" was looked at during the Study Group. This model was based on the work of Barenblatt &

Prostokishin (1993); related work has been carried out by other researchers. The basic idea of this theory for a one-dimensional object under tension is that there is a quantity ω representing damage, related to microfractures which have formed in the material and to the local (effective) modulus of elasticity (and Poisson's ratio). Thus: $\omega = 0$ for pristine material, $\omega = 1$ when it is totally smashed. This damage is increased when the material is under tension $\sigma > 0$:

$$\frac{d\omega}{dt} = q(\sigma, \omega, T)$$

for a material stretched uniformly, at temperature T. Barenblatt & Prostokishin suggest that the key quantity governing the increase of the damage is $\sigma/(1-\omega)$:

$$\frac{d\omega}{dt} = q\left(\frac{\sigma}{1-\omega}; \omega, T\right)$$

with only weak dependence upon the second and third arguments. It might then be possible to take

$$q(\sigma, \omega, T) = g\left(\frac{\sigma}{1-\omega}\right).$$

Certainly, during the Study Group, time did not allow for consideration of the direct effect of temperature on q.

Taking a simple interpretation of damage, roughly that ω is the fractional reduction in strength, so that elastic constants are given by their original (pristine) value multiplied by $(1-\omega)$,

$$\sigma \propto (1-\omega)\frac{\partial u}{\partial x},$$

with u = displacement and $\frac{\partial u}{\partial x}$ = strain

$$q(\sigma, \omega, T) = g\left(\frac{\partial u}{\partial x}\right).$$

Frémond & Nedjar, 1995, use a rate law given by the square of the strain for a higher-dimensional model for concrete damage; good comparisons between experiment and theory are claimed. In the Barenblatt & Prostokishin paper various forms of g are considered to see how damage might accumulate, these forms include powers and exponentials, or Arrhenius-type laws (allowing for sensitive dependence on strain); cut-offs are also possible, meaning that there could be some $\sigma_0 > 0$ such that $g(\sigma/(1-\omega))$ vanishes for $\sigma/(1-\omega) < \sigma_0$. Without further hard (experimental) evidence g will, for the present, be left open but it is assumed that it is non-negative: over timescales under consideration, electrodes do not repair themselves.

In Barenblatt & Prostokishin (and also the works of Frémond *et al.*), diffusion of damage is also included. The diffusion term in the Barenblatt & Prostokishin model is derived by considering the accumulation rate q to apply over a region whose extent is essentially the microstructure length scale, the grain size in the present case. This diffusion term, which vanishes when stress vanishes, then has a size inversely proportional to the square of the grain size and, apart from degenerate cases or possibly right at breaks, will then be negligible. Over the microstructure length scale, the limiting process used in obtaining diffusion is likely to be of dubious validity, and the diffusion might be better replaced by some other term, such as their original integral (averaging) one. During the Study Group, the diffusion term was then neglected but we shall return to this mechanism later.

The major difficulties with the model at this stage are the doubts over the form of the damage accumulation rate, its size (to indicate the timescale over which fractures can form and grow), and the size of stress for which it becomes significant (possibly likely to be related to the tensile strength f_t).

Simple model. As a simple illustration, a representation of the electrodes as a collection of independent rods was considered. (This model might have some real significance for a short, or fat, electrode). For the electrode to hang freely, the total force must vanish:

$$\int_0^R r\sigma \, dr = 0 \, ,$$

where $\sigma(r,t)$ = stress, in the z or vertical direction, a distance r from the axis of the electrode, taking this to be axially symmetric: R is the radius of the circle of cross-section. The net strain in the electrode of length L, $F(t)$, is related to σ and the temperature change, the latter being supposed here to be independent of z:

$$F(t) = \frac{\sigma}{E} \int_0^L \frac{dz}{1-\omega} - \theta(r) \, .$$

Here E = elastic modulus and θ, the effect of the thermal expansion, is assumed given (this is the only way in which temperature appears in this model).

Eliminating θ and F,

$$\sigma = E(F+\theta) \Big/ \int_0^L \frac{dz}{1-\omega} \, , \qquad F \int_0^R \frac{r\,dr}{\int_0^L \frac{dz}{1-\omega}} + \int_0^R \frac{r\theta\,dr}{\int_0^L \frac{dz}{1-\omega}} = 0 \, ,$$

gives a law for the increase in damage:

$$\frac{\partial \omega}{\partial t} = g \left(\frac{E}{\int_0^L \frac{d\hat{z}}{1-w}} \left(\theta(r) - \frac{\int_0^R \frac{\theta(\hat{r})\hat{r}\,d\hat{r}}{\int_0^L \frac{d\hat{z}}{1-\omega}}}{\int_0^R \frac{\hat{r}\,d\hat{r}}{\int_0^L \frac{d\hat{z}}{1-\omega}}} \right) \right) .$$

Some simulations, for problems with no z dependence, are shown in Figs. 1 - 4. Here all constants were taken as unity, $g(S) = \max(S,0)$, the material to be initially pristine ($\omega(r,0) = 0$ for all r), and $\theta = r$ (so thermal contraction is largest at the outside). It can be seen that a wave of increasing damage spreads from where θ (thermal contraction) is greatest.

(One defect in this model is that although damage localised near $r = r_0$, $z = z_0$ leads, through the decrease in stress carried around r_0, to damage increasing for other values of r, this increase is not concentrated near r_0, z_0. To get this effect, diffusivity of damage might be restored.)

Two dimensions: elasticity and damage. As in classical two-dimensional elasticity, it is convenient to employ a stress function A, such that

$$\left.\begin{aligned} \sigma_{11} = \quad & \text{stress in } x \text{ direction} && = \frac{\partial^2 A}{\partial y^2} \, , \\ \sigma_{22} = \quad & \text{stress in } y \text{ direction} && = \frac{\partial^2 A}{\partial x^2} \, , \\ \sigma_{12} = \sigma_{21} = \quad & \text{shear stress} && = -\frac{\partial^2 A}{\partial x \partial y} \, , \end{aligned}\right\} \qquad (1)$$

ensuring that the momentum equations hold (with negligible inertia).

Taking, for simplicity, elastic moduli to proportional to $(1-\omega)$,

$$\sigma_{ij} = (1-\omega) \left[\lambda(\nabla \cdot \boldsymbol{u})\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \qquad (2)$$

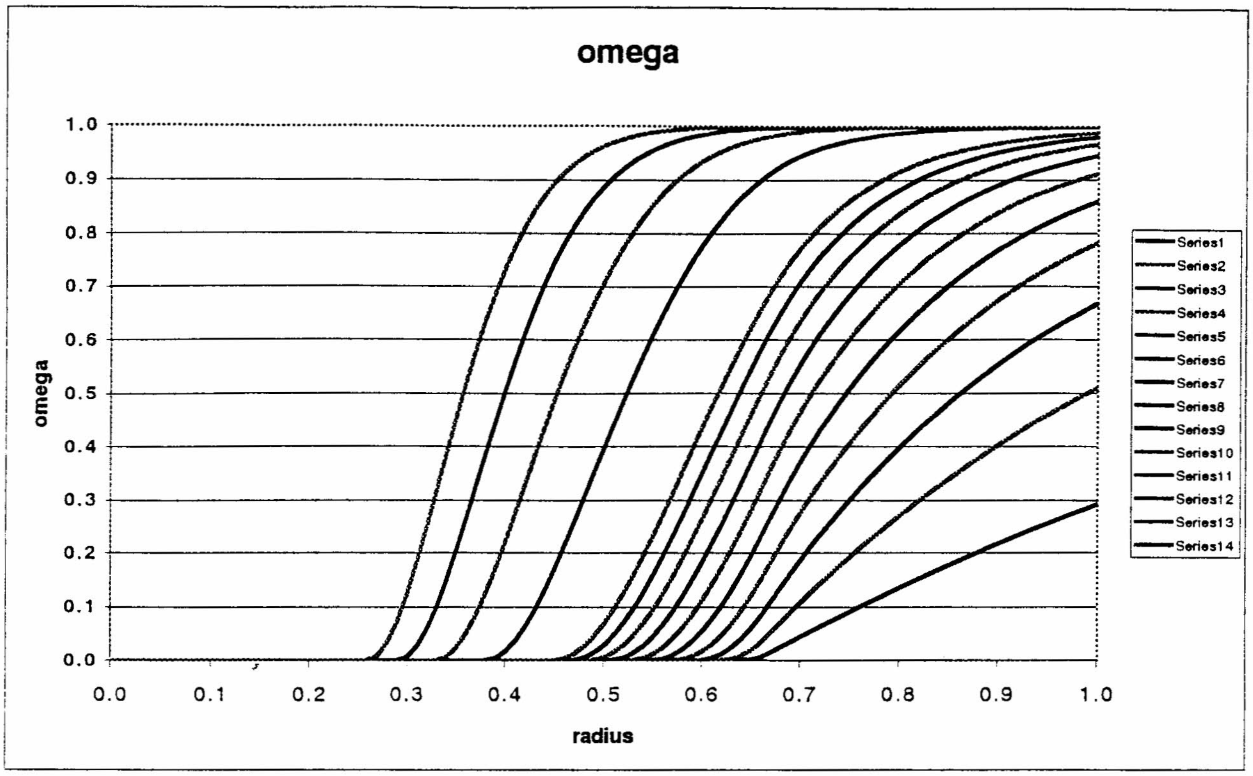

Figure 1: Damage ω against radius at times 1, 2, 3, ..., 10, 15, 20, 25, 30.

for $\boldsymbol{u} = (u_1, u_2) = (u, v) =$ displacement, $x_1 = x$, $x_2 = y$, $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 1$ for $i \neq j$. The momentum equations, (1), and elastic constitutive laws (2) are coupled with an equation for the accumulation of damage:

$$\frac{\partial \omega}{\partial t} = g\left(\frac{\sigma_{11} + \sigma_{22}}{1 - \omega}\right). \tag{3}$$

(During the Study Group, only the trace of the stress, $\sigma_{11} + \sigma_{22}$, was considered as a possibly relevant stress invariant; the determinant might possibly be used as well or instead.) Eliminating u and v, and assuming that ω and $\nabla^3 A$ are small, gives

$$\frac{\lambda + 2\mu}{\mu(\lambda + \mu)} \nabla^4 A + \frac{1}{(1 - \omega)} \left(\frac{\nabla^2 \omega \nabla^2 A}{\lambda + \mu} + \frac{1}{\mu} \left(\frac{\partial^2 \omega}{\partial x^2} \left(\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 A}{\partial y^2} \right) \right.\right.$$
$$\left.\left. + \frac{\partial^2 \omega}{\partial y^2} \left(\frac{\partial^2 A}{\partial y^2} - \frac{\partial^2 A}{\partial x^2} \right) + 4 \frac{\partial^2 \omega}{\partial x \partial y} \frac{\partial^2 A}{\partial x \partial y} \right) \right) = O(\omega^2, \omega |\nabla^3 A|).$$

Looking for a simple perturbation solution,

$$A \sim x^2/2 + A_1 + \dots, \quad \omega \ll 1,$$

and taking $g(S) \sim S - 1$ for $S \to 1+$, gives

$$(\lambda + 2\mu) \nabla^4 A_1 + \mu \nabla^2 \omega + (\lambda + \mu) \left(\frac{\partial^2 \omega}{\partial x^2} - \frac{\partial^2 \omega}{\partial y^2} \right) = 0,$$

to leading order, *i.e.*

$$\nabla^4 A_1 + \frac{\partial^2 \omega}{\partial x^2} - \frac{\lambda}{\lambda + 2\mu} \frac{\partial^2 \omega}{\partial y^2} = 0, \tag{4}$$

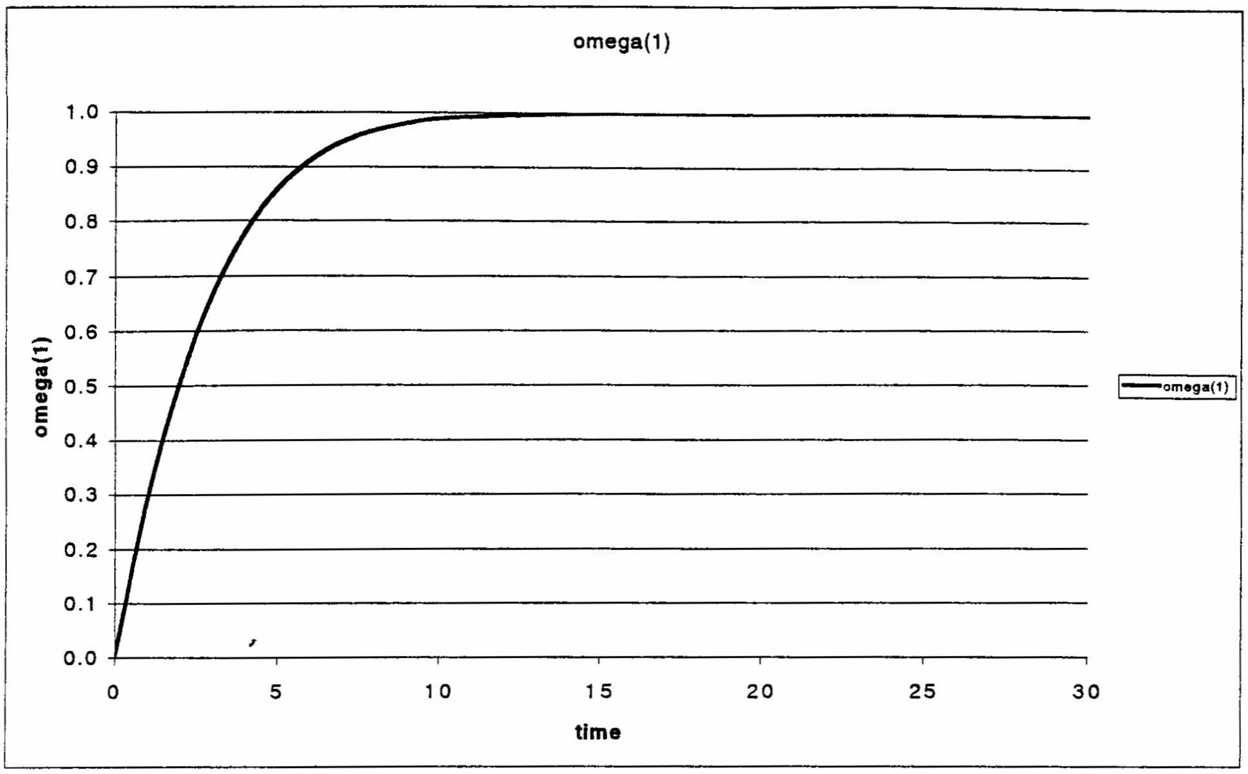

Figure 2: Plot of ω at radius $r = 1$ against time.

with

$$\frac{\partial \omega}{\partial t} = \omega + \nabla^2 A_1 \, . \tag{5}$$

Ideally, (4) and (5) should be solved subject to appropriate boundary conditions for A_1 (probably growth conditions, if this analysis is thought of as applying to some local problem). Three simple types of solution are

(a) sinusoidal:

$$\omega = \mathrm{Re}\{e^{i(\alpha x + \beta y) + \gamma t}\}, \ \nabla^2 A_1 = \mathrm{Re}\left\{ \frac{\lambda \beta^2 / (\lambda + 2\mu) - \alpha^2}{\alpha^2 + \beta^2} e^{i(\alpha x + \beta y) + \gamma t} \right\} ,$$

$$\gamma = \frac{\beta^2}{\alpha^2 + \beta^2} \left(1 + \frac{\lambda}{\lambda^2 + 2\mu} \right) > 0 ;$$

(b) no x dependence:

$$\frac{\partial^4 A_1}{\partial y^4} = \frac{\lambda}{\lambda + 2_\mu} \frac{\partial^2 \omega}{\partial y^2} , \quad \frac{\partial \omega}{\partial t} = \frac{\partial^2 A_1}{\partial y^2} + \omega ,$$

$$\text{so } \frac{\partial^5 A_1}{\partial t \, \partial y^4} = \left(\frac{\lambda}{\lambda + 2\mu} + 1 \right) \frac{\partial^4 A_1}{\partial y^4} ;$$

(c) no y dependence:

$$\frac{\partial^4 A_1}{\partial x^4} = - \frac{\partial^2 \omega}{\partial x^2} , \quad \frac{\partial \omega}{\partial t} = \frac{\partial^2 A_1}{\partial x^2} + \omega \quad \text{so } \frac{\partial^5 A_1}{\partial t \partial x^4} = 0 .$$

These cases all indicate linear instability and increase of damage.

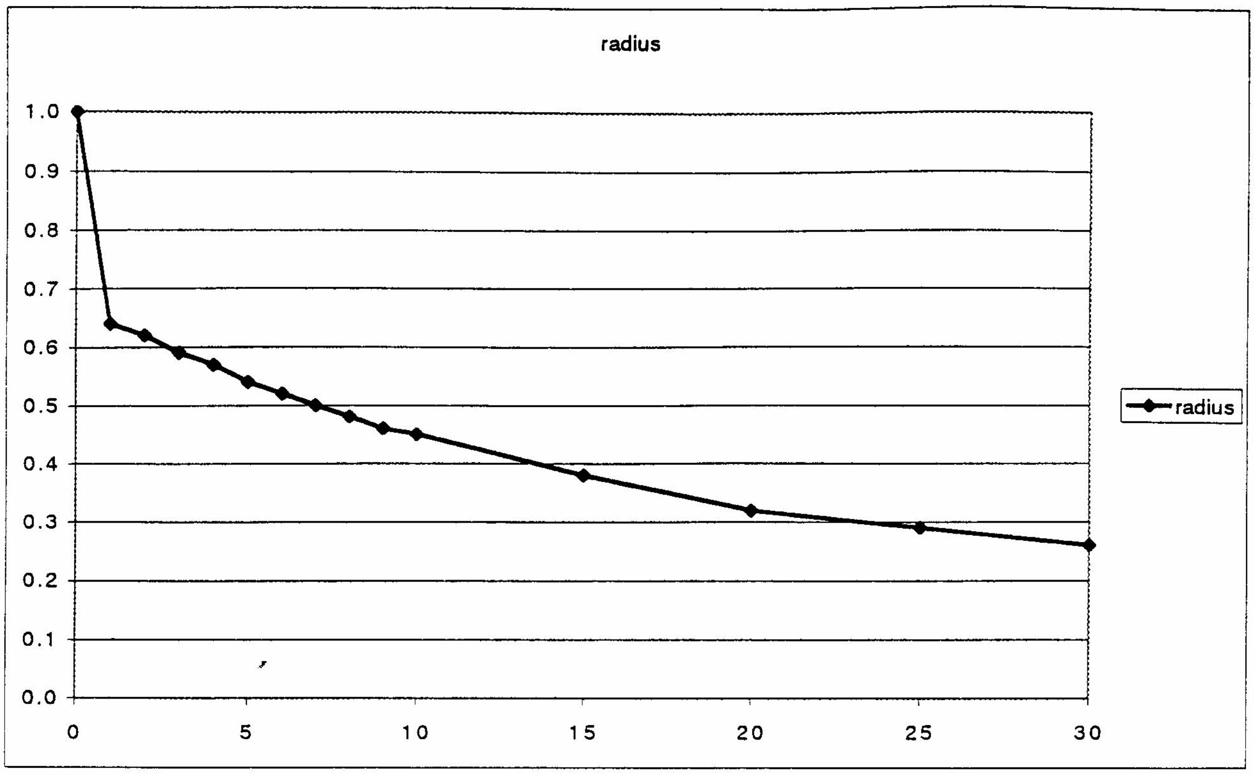

Figure 3: Plot of damage penetration front (interface between $\omega = 0$ and $\omega > 0$) against time.

Special Cases.

(1) Reduction to the one-dimensional model

Taking a two-dimensional problem, with the material occupying a region $|x| < L$, $|y| < h$, $h/L = \varepsilon \ll 1$, and taking stress-free boundary conditions on the sides, $\sigma_{12} = \sigma_{21} = 0$ on $y = \pm h$. Displacement and stresses could be sought as power series in ε. For instance, the displacement in the x direction is given in terms of

$$\boldsymbol{u} \sim \boldsymbol{u}_0 + \epsilon^2 \boldsymbol{u}_1 + \dots \text{ for } \epsilon \to 0 .$$

Substituting such an expression into the two-dimensional equations, and using the boundary conditions (which may be regarded as compatability conditions) eventually yields

$$u_0 = u_0(x,t) \qquad \text{(the leading term for } x \text{ displacement)},$$

$$v_0 = -\frac{\lambda}{\lambda + 2\mu}\frac{\partial u_0}{\partial x} y \qquad \text{(the leading term for } y \text{ displacement)},$$

$$\frac{\partial^2 A_0}{\partial y^2} = (1 - \omega_0)\,\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu}\frac{\partial u_0}{\partial x},$$

and (to get $\sigma_{12} = -\frac{\partial^2 A}{\partial x \partial y} = 0$ on the boundaries),

$$\frac{\partial}{\partial x}\left((1 - \omega)\,\frac{\partial u_0}{\partial x}\right) = 0 .$$

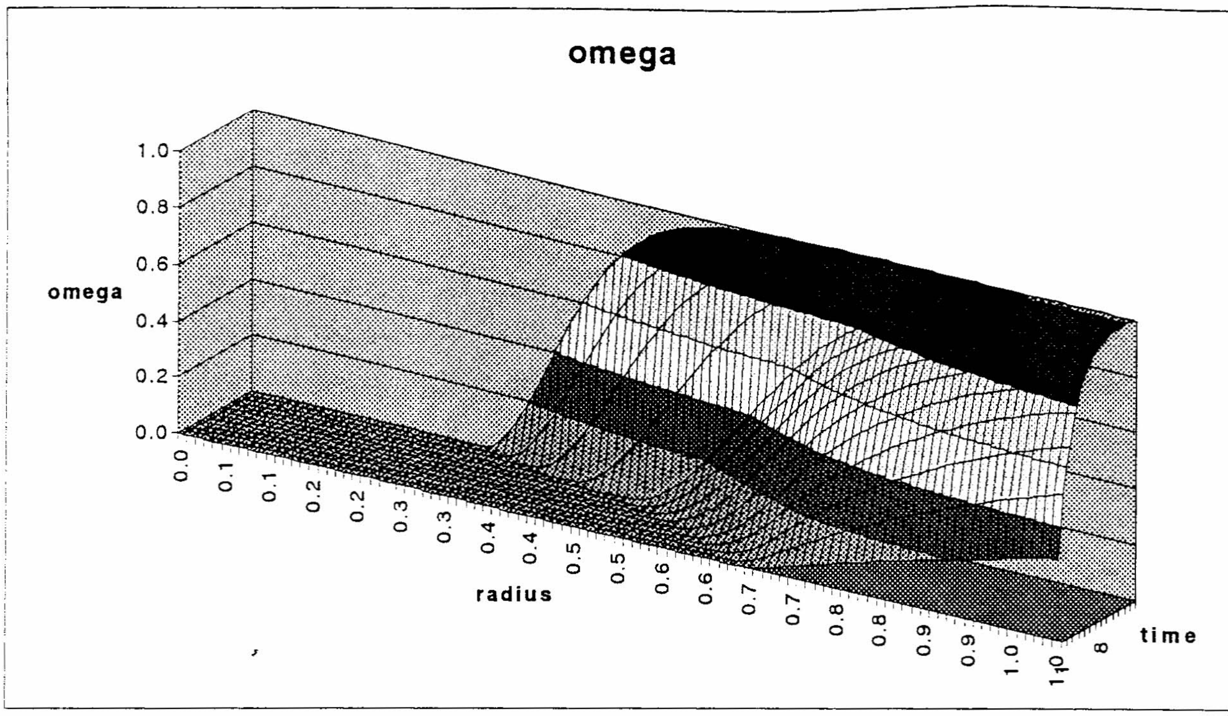

Figure 4: Three-dimensional plot of damage against position (radius) and time.

This means that, to leading order,

$$\sigma = \frac{\partial^2 A}{\partial y^2} = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}(1-\omega)\frac{\partial u_0}{\partial x} = \sigma(t)$$

and the Barenblatt & Prostokishin model is recovered.

(2) Cracks

Geometry with the y length scale much less than the x scale (and the z one) might also be relevant to both the initiation and extension of fractures. This was not taken further during the Study Group but it was speculated that crack-like travelling-wave solutions might (see 4 below) be related to the "Fictitious-Crack Model" (Hillerborg, Hillerborg *et al.*).

A situation like in Fig. 5 could be envisaged: on the crack (say $y = 0$, $x < Vt$) the material is broken: $\omega \equiv 1$. Near the crack tip there is fracturing: $\omega = O(1)$ and $\frac{\partial \omega}{\partial t} = O(1)$ for $|y| \ll 1$, $|x - Vt| \ll 1$ because σ is large here. Near the crack but away from tip the material is damaged but not subject to further damage: $\omega = O(1)$ but $\frac{\partial \omega}{\partial t} \ll 1$ for $|y| \ll 1$, $Vt - x = O(1)$ due to the release of stress from the neighbouring break on $y = 0$.

(3) Radial symmetry in two dimensions

With displacements, u, only in the radial, r, direction, a radially symmetric stress function $A = A(r,t)$ and damage $\omega = \omega(r,t)$ satisfy

$$\sigma_{rr} = \text{radial stress} = \frac{1}{r}\frac{\partial A}{\partial r}, \qquad \sigma_{\theta\theta} = \text{hoop stress} = \frac{\partial^2 A}{\partial r^2}, \qquad \sigma_{r\theta} = \text{shear stress} = 0\,,$$

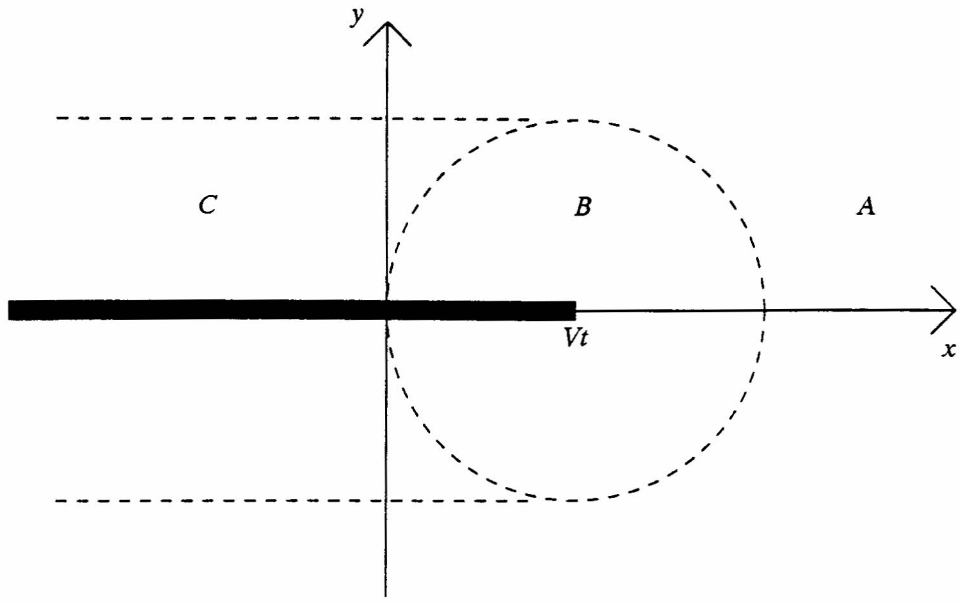

Figure 5: Damage near a crack. A: stress, σ, damage, ω, and rate of change of damage, $\partial\omega/\partial t$, all small. B: σ, ω, $\partial\omega/\partial t$ all significant. C: σ and $\partial\omega/\partial t$ small but ω significant.

$$E(1-\omega)\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial A}{\partial r} - \nu\frac{\partial^2 A}{\partial r^2} \quad \text{and} \quad E(1-\omega)u = r\left(\frac{\partial^2 A}{\partial r^2} - \frac{\nu}{r}\frac{\partial A}{\partial r}\right),$$

(Timoshenko & Goodier) for Poisson's ratio ν, with

$$\frac{\partial\omega}{\partial t} = g\left(\frac{\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r}}{1-\omega}\right),$$

again taking a damage-accumulation rate dependent on $\text{tr}(\sigma)/(1-\omega)$. This has yet to be taken further, except that solutions of approximately travelling-wave type were briefly considered: see *(4)*.

(4) Travelling-waves and approximate travelling-waves

Two types of travelling-wave solutions are considered here: the first is planar and assumes only small changes in the stress; the second is radial or axisymmetric and involves solving the linear equations of stress. This second type of solution is perhaps of more interest in terms of the hanging electrode, since failure is thought to begin inside the cylindrical electrode and spread outward.

Considering first the planar solution, let us recall the Barenblatt & Prostokishin form of the equation for damage propagation with $g(S) = S$:

$$\frac{\partial\omega}{\partial t} = q(\omega,\sigma;T) = \frac{\sigma}{1-\omega} \quad (6)$$

where again $\omega(\boldsymbol{x},t)$ measures the damage at $(\boldsymbol{x},t)$, and σ represents the local stress. In the undamaged, pristine material, $\sigma = \sigma_0$. In Barenblatt & Prostokishin, σ_0 was simply a constant because their analysis was one-dimensional. For the present planar analysis, $\sigma(t)$ is a measure of the local stress tensor at time t, and $\sigma_0(t)$ is its value in the pristine material. So σ_0 is assumed to be independent of position x, or at least, slowly varying, and we assume that $\sigma \simeq \sigma_0$. This is certainly not the case near the region of failure ($\omega = 1$) where $\sigma \searrow 0$, but it is a basic description of the interface between the pristine region ($\omega = 0$) and the damaged region ($\omega > 0$). Here one would expect σ_0 to increase in time as the load is supported by a decreasing volume of material, but to remain spatially uniform, at least to lowest order.

Now consider the case of a planar damage front moving into a pristine region (cf. Fig. 6). Let

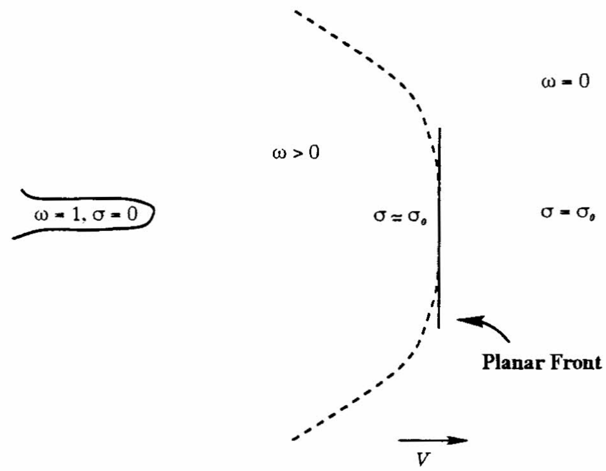

Figure 6: Schematic of a damage front moving into the pristine region.

x be the coordinate perpendicular to the front, and let $x = p(t)$ be the position of the front, and $V(t) = p'(t)$, its velocity. Define $\xi = x - p(t)$ to be the distance from the front at time t. We wish to look for a solution $\omega(\xi)$ of (6) which is valid at least near $\xi = 0$ (near the front). Since

$$\frac{\partial \omega}{\partial t}(\xi(x,t)) \quad = \quad \omega'(\xi)(-V(t)) \ ,$$

the kinetic equation (6) then becomes

$$(1-\omega)\omega'(\xi) \quad = \quad -\frac{\sigma_0}{V} \tag{7}$$

where as usual $'$ indicates differentiation with respect to a single variable. In general (7) would be difficult to solve since its right-hand side is some unknown (and perhaps complicated) function of t. But it is reasonable to look for solutions where the stress σ_0 and the velocity V grow at the same rate (*i.e.*, depend on time in the same way). In such cases, the right-hand side of (7) would be independent of time, and it is possible to solve (7) and obtain an expression for the form of the damage function $\omega(x,t)$ near the front:

$$\omega(x,t) \quad = \quad \begin{cases} 1 - \sqrt{1 + 2\frac{\sigma_0}{V}(x - p(t))} & x < p(t) \\ 0 & x \geq p(t) \end{cases} . \tag{8}$$

(If the region where most of the damage occurs, *i.e.* the zone where (8) is important is small, it is reasonable to regard this wave as quasi-steady so time variation of V can be neglected.)

The above calculation does not prove the existence of a planar travelling wave; rather it suggests that it is possible for a damaged region ($\omega > 0$) to spread into a pristine region ($\omega = 0$) at a speed V related to the rate at which damage accumulates ahead of the front. More precisely, it indicates that such a situation is consistent with the damage-kinetics equation (6). Furthermore, the representation for ω given in (8) is valid only near the plane $x = p(t)$.

Now we turn our attention to the question of axisymmetric travelling waves moving outward in the radial direction. Let A again be the stress function, and assume that both this stress function and the damage are independent of the angle θ: $A = A(r,t)$ and $\omega = \omega(r,t)$. Then by definition, the relationship between the stress function and stress itself is, as before,

$$\sigma_{rr} = \frac{1}{r}\frac{\partial A}{\partial r}, \qquad \sigma_{\theta\theta} = \frac{\partial^2 A}{\partial r^2}, \qquad \sigma_{r\theta} = 0.$$

Assuming that the numerator of the right-hand side of (6) can be interpreted as the trace of the stress tensor, then this kinetic equation for damage, (6), becomes

$$\frac{\partial \omega}{\partial t} = q(\omega, \sigma) = \frac{\sigma_{rr} + \sigma_{\theta\theta}}{1-\omega}. \tag{10}$$

Now suppose that the travelling wave is moving with constant speed V and is located at $r = Vt$. Define $\epsilon\zeta = r - Vt$ to be the distance from the front where $\epsilon \ll 1$ implies that we are near the front. From the equations of elasticity, one can then derive that

$$\frac{\partial}{\partial \zeta}\left(\frac{1}{1-\omega}\frac{\partial^2 A}{\partial \zeta^2}\right) \simeq 0. \tag{11}$$

Here and in what follows, the symbol $\simeq$ implies that only the lowest-order terms in ϵ are considered. In addition, the present kinetics equation (10) can be written as

$$-V\frac{\partial \omega}{\partial \zeta} \simeq \frac{1}{1-\omega}\frac{\partial^2 A}{\partial \zeta^2}. \tag{12}$$

Combining (11) and (12), one obtains a system of two equations for A and ω:

$$\frac{\partial^2 A}{\partial \zeta^2} \simeq C(t)(1-\omega),$$
$$V\frac{\partial \omega}{\partial \zeta} \simeq -C(t),$$

where C is independent of ζ but may depend on t. It may be noted that this two-dimensional travelling-wave solution differs markedly from the one-dimensional one, (7), with the factor $(1-\omega)$ disappearing, indicating the importance of the geometry.

5 Discussion and Further Work

(i) Damage production. One possible problem with the damage model of §4 is the prospect of a conflict with measurement leading to the fracture energy. Elkem's experiments showed that the energy involved in forming a macroscopic break is roughly proportional to the square of the length. (Ideally, energy $\propto G_F L^2$ for a length scale L. They actually find energy increases rather more slowly, possibly due to damage near the points where forces are applied.) The sort of damage model discussed in the previous section would be expected to lead to energy $\propto L^3$. It is conceivable that the extension zone of a narrow damage zone might, with suitable laws for damage accumulation (perhaps with Arrhenius or exporential dependence on stress), allow energy to be like L^2. (Given a rate-dependent damage accumulation, it might also be expected that the energy input to produce a break should depend on how the break is formed: larger pores and stresses – faster fracture – should change G_F.)

Of course to make any concrete prediction for this damage model, it is necessary to know q: what components of σ it depends on; its qualitative form (quadratic, exponential, ...); its size; sizes of σ for which $\frac{\partial \omega}{\partial t}$ is significant. (Regarding the first point, it should also be noted that the damage itself might not be isotropic, Kachanov (1980).) A tentative, but simple, experiment might involve stretching a uniform bar, with either controlled tension or controlled stretch. For example, assuming that $q = g(\sigma/(1-\omega))$, $\sigma = E(1-\omega)\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial x}$ = given strain = C, (elastic modulus E = constant),

$$\frac{\partial \omega}{\partial t} = g(CE)$$

and, assuming an initially pristine material,

$$\omega = g(CE)t\,, \quad \sigma = CE(1 - g(CE)t).$$

This could be used to determine the function g. Acoustic measurements could be carried out simultaneously to get an indication of the actual rate of damage accumulation (the distribution of the sound levels from microfracture might give some extra information about the mechanisim of fracture, as in geomechanics (Cox & Meredith); there could conceivably be some fractal structure). Such a simple experiment is likely to be badly affected by any initial damage, unlike the experiment used by Elkem. Taking σ = constant and specified (as Barenblatt & Prostokishin; see also Kachanov, 1961) with, say, a simple power-law damage accumulation rate,

$$\frac{\partial \omega}{\partial t} = g\left(\frac{\sigma}{1-\omega}\right) = \frac{A\sigma^\alpha}{(1-\omega)^\alpha} \qquad (a > 0,\ \alpha \geq 0),$$

$$\omega = 1 - ((1-\omega_0(x)) - A(\alpha+1)\sigma^\alpha t)^{1/(\alpha+1)}\,,$$

with ω_0 = initial damage. For a pristine bar of length L, its extension is then

$$u_1 = L\frac{\partial u}{\partial x} = \frac{\sigma}{E(1-\omega)} = \frac{\sigma}{E}(1 - A(\alpha+1)\sigma^\alpha t)^{-1/(\alpha+1)}. \tag{14}$$

More generally

$$u_1 = \int_0^L \frac{\partial u}{\partial x}dx = \frac{\sigma}{E}\int_0^L \left((1-\omega_0(x))^{\alpha+1} - A(\alpha+1)\sigma^\alpha t\right)^{-1/(\alpha+1)} dx\,. \tag{15}$$

Rather than the inverse power law, $u_1 \propto (t^* - t)^{-1/(\alpha+1)}$ for a break time t^*, given by (14), the more general case, as (15), assuming that

$$\omega_0(x) \sim a - b(x - x_0)^2 \ldots \text{ for } x \to x_0\,,$$

with $a = \max\{\omega_0\} > 0$ and $b > 0$, gives $u_1 \propto (t^* - t)^{-(1-\alpha)/2(1+\alpha)}$ as $t \to t^*-$ for $\alpha < 1$ but u_1 bounded as $t \to t^*-$ for $\alpha > 1$. The special case, $\alpha = 1$, is expected to give logarithimic growth of the net extension.