

Design of moulded valve

Domino Printing Sciences

1 Background

Domino wish to make a small valve by moulding an aperture in a rubber block. The valve is to be closed by compression from above, and opened by reducing the compression. The problem presented to the study group was to design a shape for the valve such that it closes from the sides inwards rather than outwards from the centre. This is required in order to prevent the production of satellite droplets that would spoil the printed image. An additional requirement is that the moulded shape of aperture should be smooth (without sharp edges) so that the former can be easily machined. In order to achieve the longest operational lifetime of the valve the rubber must always be under compression and stress concentrations should be minimised. Unfortunately a smooth stress distribution and a smooth geometry are incompatible, since an aperture with rounded corners will necessarily have a stress singularity at the corner.

In the first part of this report we shall consider the valve as a small aperture in an infinite rubber sheet undergoing linear plane strain. In section two we shall discuss how the nonlinearity of the rubber will affect our solutions in section one. Finally, in section three, we shall discuss the three dimensional problem.

2 Linear plane strain analysis

The aspect ratio (length:height) of the fully open valve is of order 10:1, so the strain of the material required to close the valve is of order 10%. This small strain allows a number of approximations to be made. First, the rubber can be treated as a linear elastic material (which is virtually incompressible). Second, since the height of the aperture is small compared to its width, the boundary conditions at the surface of the aperture may be applied (at leading order) along the centre-line of the aperture (*i.e.* the aperture is treated as having negligible height). We shall also assume that the strain is confined to the plane perpendicular to the direction of fluid flow (see figure 1). Finally, we note that the pressure required to expel the fluid within the valve (of order 100 Pa) is small compared with the elastic stress required to close the valve (of order 50 kPa) and so fluid pressures within the valve may be neglected.

With these approximations the aperture may be modelled as a thin line crack along $y = 0$ for $|x| \leq a$ subject to a uniaxial compressive stress at infinity. This is equivalent to the same crack subject to a (negative) internal pressure p (see figure 2). The boundary

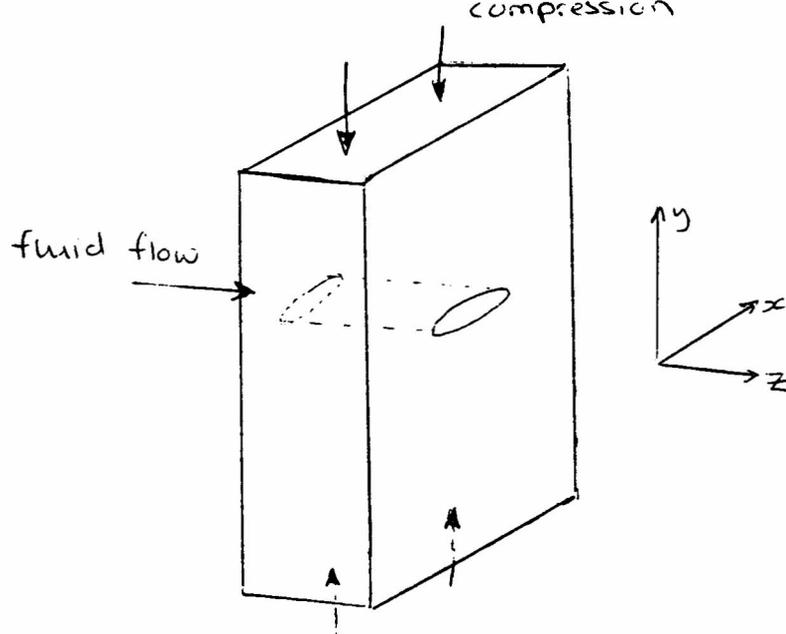


Figure 1: Sketch of the valve geometry

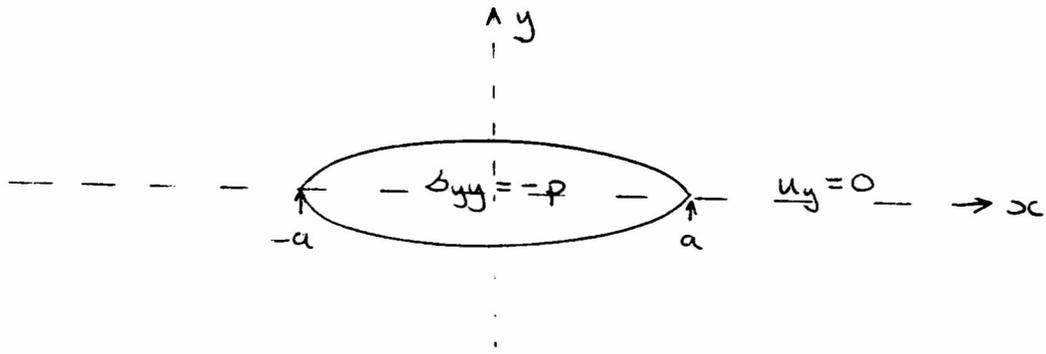


Figure 2: Geometry of the crack-closing problem.

conditions along the x -axis are that the vertical component of stress $\sigma_{yy} = -p$ within the aperture ($|x| < a$), and that the vertical displacement $u_y = 0$ outside the aperture ($|x| \geq a$).

This mixed boundary value problem is a standard problem in the linear elasticity of thin cracks (see, for example, Green and Zerna, "Theoretical Elasticity" (Oxford)) and the vertical displacement u_y along $y = \pm 0$ is given by

$$(1) \quad u_y = \pm \frac{p}{2\mu} \sqrt{a^2 - x^2} \quad \text{for } |x| \leq a.$$

Therefore if the initial shape h of the aperture is chosen to be an ellipse

$$h = \frac{p_0}{2\mu} \sqrt{a^2 - x^2},$$

the aperture will close simultaneously along its whole length when the pressure inside is reduced to $-p_0$. The stress along $y = 0$ is given by

$$\sigma_{yy} = -p \left(\frac{|x|}{\sqrt{x^2 - a^2}} - 1 \right) \quad \text{for } |x| \geq a.$$

This has a square root singularity at $x = \pm a$, but the rubber is always under compression.

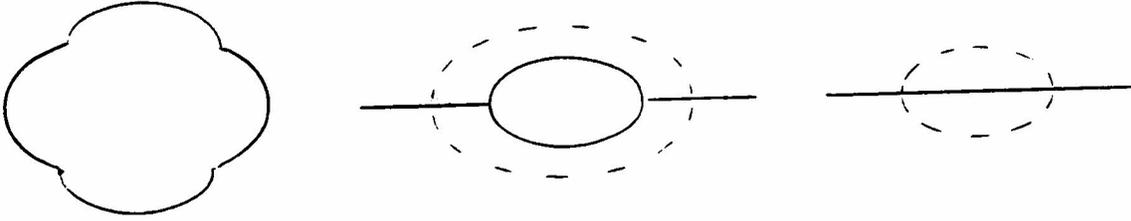


Figure 3: Sketch of the two stage collapse of the ‘flying saucer’. Reducing the internal pressure to $-p_0$ collapses the outer part of the aperture, forming a inner elliptical aperture. A further reduction in pressure to $-p_0 - p_1$ collapses this inner ellipse.

2.1 The ‘Flying Saucer’

What is required by Domino is an aperture that closes from the edges inwards. To illustrate how this can be achieved we consider the following ‘thought experiment’. Consider an aperture of undeformed shape

$$(2) \quad h = \begin{cases} 0 & |x| > a_0, \\ \frac{p_0}{2\mu} \sqrt{a_0^2 - x^2} & a_1 < |x| \leq a_0, \\ \frac{p_0}{2\mu} \sqrt{a_0^2 - x^2} + \frac{p_1}{2\mu} \sqrt{a_1^2 - x^2} & |x| \leq a_1 \end{cases}$$

formed by adding together two ellipses. Reducing the pressure inside the aperture to $-p_0$ causes a vertical displacement,

$$u_y = -\frac{p_0}{2\mu} \sqrt{a^2 - x^2}$$

along the line $y = +0$ for $|x| \leq a_0$. This displacement closes the aperture between a_0 and a_1 leaving a smaller elliptical hole for $|x| < a_1$, (see figure 3). As the pressure is further reduced, the closed section of the original aperture ($a_1 < |x| < a_0$) is under compression with zero shear stress and behaves like uncut material, so that the boundary condition along this section of the y -axis becomes $u_y = 0$ for all additional reductions in pressure). Hence the subsequent displacements will be those for an aperture of width a_1 .

2.2 Calculation of Aperture Shape

We now consider an aperture whose contact point moves smoothly inwards as the pressure is reduced (so that the crack ‘zips’ closed from the sides inwards). The half-width of the aperture a is now a continuous (and increasing) function of the interior pressure p . At $p = 0$ the rubber is undeformed and $a = a_0$ the maximum width of the aperture. The aperture is fully closed when the pressure inside is equal to $-p_0$. When the pressure

within the opening is equal to p , the aperture is closed for $|x| \geq a(p)$ and open for $|x| < a(p)$. Thus the rate of change of displacement with pressure is given by

$$(3) \quad \frac{du}{dp} = \begin{cases} \frac{1}{2\mu} \sqrt{a^2(p) - x^2} & \text{for } |x| < a(p), \\ 0 & \text{for } |x| \geq a(p) \end{cases}$$

with the initial condition $u = 0$ at $p = 0$. Integrating this equation we obtain the following integral expression for the displacement when the aperture is fully closed,

$$(4) \quad u(x) = -\frac{1}{2\mu} \int_x^{a_0} \sqrt{a^2 - x^2} \frac{dp}{da} da$$

Therefore the initial shape of the aperture $h(x)$ should be chosen as $-u(x)$. By choosing an appropriate function $p(a)$ we can calculate the displacement and hence the initial shape of the aperture. In this way we can tailor the closing behaviour of the aperture. A few simple examples are given below and are illustrated in figure 4.

1. $\frac{dp}{da} = p_0 \delta(a - a_0)$ (where δ is the Dirac delta function) recovers the original ellipse solution (equation 1).
2. $\frac{dp}{da} = \frac{p_0}{a_0}$ produces an aperture of shape

$$h(x) = \frac{p_0}{4\mu} \left(\sqrt{a_0^2 - x^2} - \frac{x^2}{a_0} \cosh^{-1} \left(\frac{a_0}{|x|} \right) \right)$$

that closes at a constant rate with decreasing pressure.

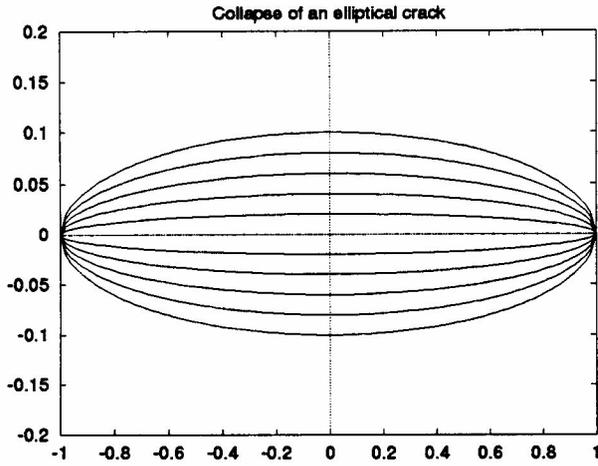
3. $\frac{dp}{da} = \frac{2p_0}{a_0^2} a$ gives an aperture with a rate of closing with decreasing pressure proportional to the current width of the aperture. Thus if the pressure is reduced at a uniform rate in time, the aperture closes slowly at first and then slams shut. The shape of the aperture is given by

$$(5) \quad h(x) = \frac{p_0}{3\mu a_0^2} (a_0^2 - x^2)^{3/2}$$

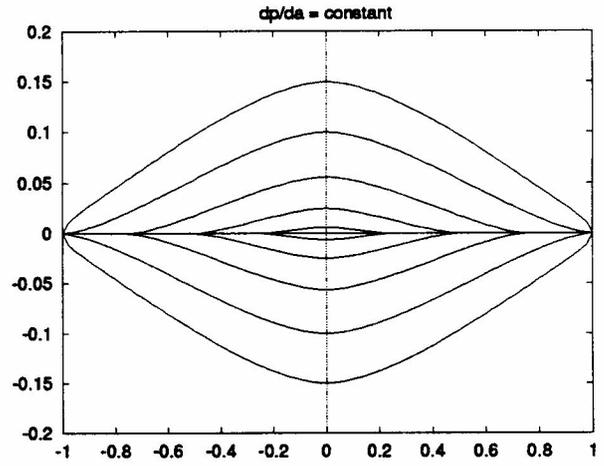
4. $\frac{dp}{da} = \frac{2p_0}{a_0^2} (a_0 - a)$ produces an aperture with the opposite closing behaviour to the previous example. When a is close to a_0 there is large change in aperture with pressure, and the rate of closing decreases with the size of the aperture. The shape of the aperture is given by

$$(6) \quad h(x) = \frac{p_0}{\mu} \left(\frac{1}{2} \sqrt{a_0^2 - x^2} - \frac{x^2}{2a_0} \cosh^{-1} \left(\frac{a_0}{|x|} \right) - \frac{1}{3a_0^2} (a_0^2 - x^2)^{3/2} \right)$$

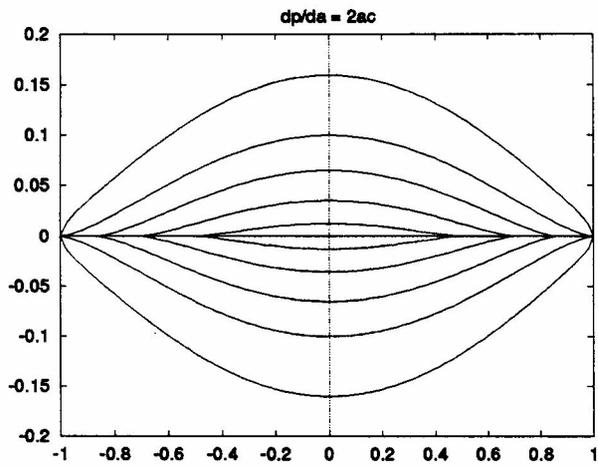
and is more peaked in the centre than the other examples.



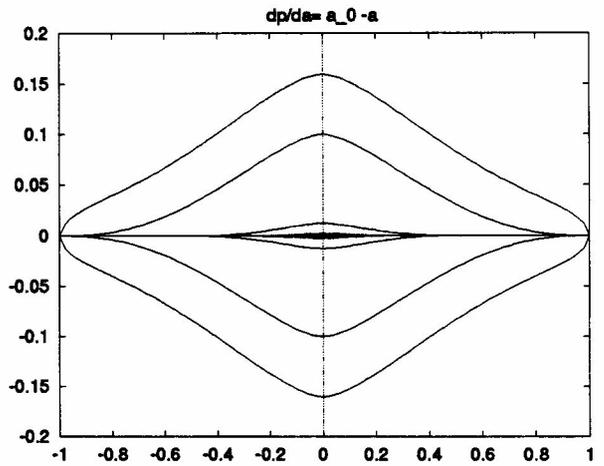
(a) $dp/da = p_0 \delta(a - a_0)$



(b) $dp/da = p_0/a_0$



(c) $dp/da = 2p_0 a/a_0^2$



(d) $dp/da = 2p_0(a_0 - a)/a_0^2$

Figure 4: Plots of the aperture shape for different dp/da . The four inner curves in each figure show the shape of the aperture at $p = 0$, $-\frac{1}{4}p_0$, $-\frac{1}{2}p_0$ and $-\frac{3}{4}p_0$. The outer curve shows the shape with an added ellipse.

All the last three shapes have cusps at the edges. This is unavoidable, since the stress must be non-singular at the moving edge. In examples 2 and 3, h scales as $(x - a_0)^{3/2}$ in the neighbourhood of a_0 giving a stress with a discontinuous gradient at a_0 . In example 4, the cusp is sharper, scaling as $(x - a_0)^{5/2}$, but has the advantage of having a stress with a continuous gradient. However, for ease of moulding an elliptical displacement could be added as shown in figure 4. In operation the valve should always be held under sufficient compression to squash the corners into cusps. This, of course, introduces a singularity in the stress at $x = \pm a_0$. However, this is a static compressive stress that shouldn't restrict the lifetime of the valve too severely.

It is important to realise that while the rubber is in compression, the problem is quite different from classical crack theory. The term 'crack' is somewhat misleading in this context. In fact it is a contact problem with free boundaries dividing contact regions from free regions. This is why there is no stress singularity at the ends of the opening.

2.3 The Inverse Problem

In order to solve the inverse problem of finding the closing behaviour of a given shape we need to invert the integral equation

$$(7) \quad h(x) = \frac{1}{2\mu} \int_x^{a_0} \sqrt{a^2 - x^2} \frac{dp}{da} da$$

Differentiating this equation with respect to x gives

$$\frac{dh}{dx} = -\frac{x}{2\mu} \int_x^{a_0} \frac{dp/da}{\sqrt{a^2 - x^2}} da$$

which is an Abel-type integral transform. After some manipulation, it can be shown that the inverse of this transform is

$$(8) \quad p(a) = -\frac{4\mu}{\pi} \int_a^{a_0} \frac{dh/dx}{\sqrt{x^2 - a^2}} dx$$

where we have used the boundary condition $p(a_0) = 0$.

A given shape $h(x)$ will have the derived closing behaviour, from the ends inwards, if $p(a)$ as derived from (8) is monotonic. As a simple example, consider a parabolic aperture, with $h(x) = k(a_0^2 - x^2)$. This has a pressure width dependence of

$$p(a) = -\frac{8\mu k}{\pi} \sqrt{a_0^2 - a^2}$$

3 Large strain behaviour

The previous analysis assumes small strain and linear elasticity. An elliptical aperture in a linear elastic material still collapses instantaneously along its length even when the

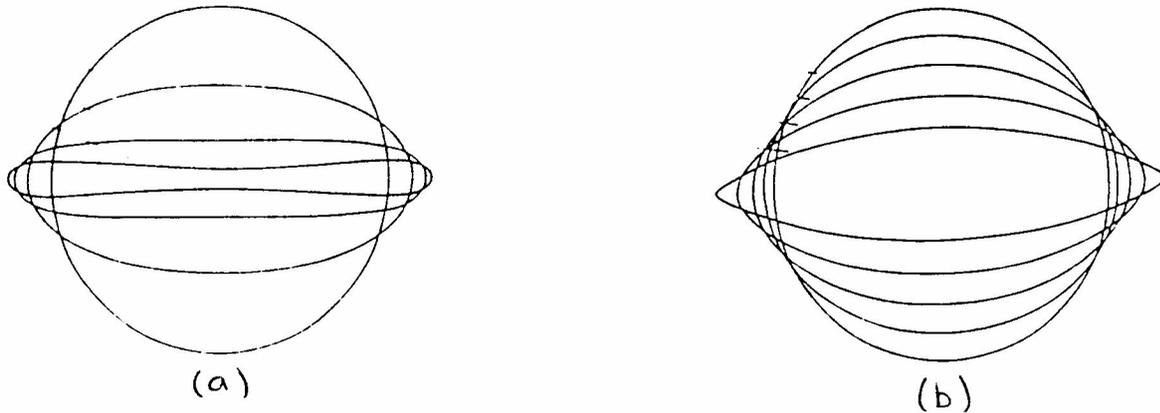


Figure 5: Sketch of the closing of a circular hole in (a) a strain-softening, and (b) a strain-hardening material (taken from Varley and Cumberbatch).

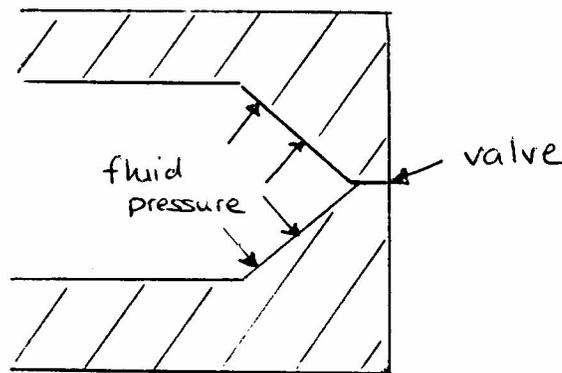


Figure 6: Sketch of the full moulding showing the position of the valve.

strains are not small. The only difference from the small strain behaviour is that the width of the aperture increases as the height is reduced.

Rubber is known to be a nonlinear elastic material. The effect of nonlinear elasticity on the collapse of a circular aperture is considered by Varley and Cumberbatch (1980), *Journal of Elasticity*, vol 10, pages 341-405. A strain-hardening material collapses at the edges first (where the strains are smallest), while a strain-softening material collapses at the centre first (see figure 5).

As the strains in the rubber are only of order 10%, we do not believe these large strain effects will change the closing behaviour, except for the case of an elliptical aperture where nonlinear elasticity determines whether it closes at the centre of edge first.

4 Three dimensional problem

The shape of the full moulding is sketched in figure 6. A numerical computation is required to calculate the detailed elastic deformation of the moulding. The problem may

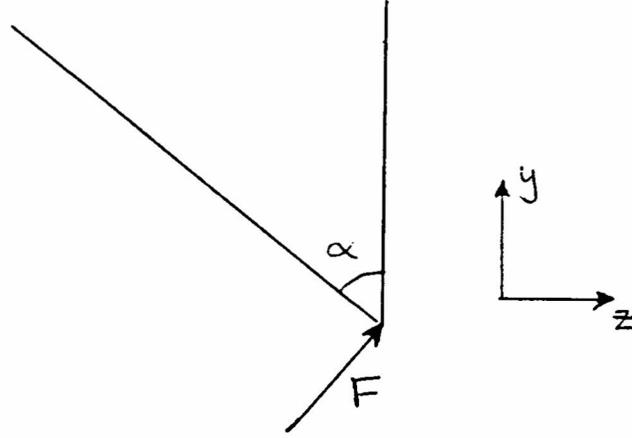


Figure 7: A simplified model of the opening of the valve by fluid pressure. The moulding is modelled as a wedge of angle α subject to a point force F at the tip.

be written in variational form as minimising the strain energy

$$\int_V \sigma_{ij} e_{ij} dV$$

over all displacements satisfying the constraint of no interpenetration of surfaces and appropriate boundary conditions along the free and contacting parts of the boundary. Implementation via finite elements is standard, see for example Elliott and Ockendon, “Weak and Variational Methods for Free and Moving Boundary Problems” (Pitman) although there are several other books that treat the elastic contact problem in detail.

To get an indication of the large scale behaviour of the moulding, we consider the simplified problem of a point force acting at the tip of a wedge of angle α (see figure 7). The solution is given in Green and Zerna, “Theoretical Elasticity” (Oxford) page 258. The displacement of the tip has a logarithmic singularity so that as $r \rightarrow \infty$

$$u_y \sim \frac{F \log(r)}{4\mu} \sin(\alpha) \left(\frac{1}{\alpha + \sin(\alpha)} + \frac{1}{\alpha - \sin(\alpha)} \right)$$

$$u_z \sim \frac{F \log(r)}{2\mu} \left(\frac{\cos^2(\alpha/2)}{\alpha - \sin(\alpha)} - \frac{\sin^2(\alpha/2)}{\alpha + \sin(\alpha)} \right)$$

Both u_y and u_z decrease with increasing α , however the ratio u_y/u_z increases with α and so in order to ensure that the displacement of the tip is as near vertical as possible, α should be chosen to be large.

List of Participants

The study group participants who worked on this problem were, Richard Craster, Ellis Cumberbatch, Richard Day, Barbera van de Fliert, Oliver Harlen, John Lister, Stefan Llewellyn Smith and Taryn Malcolm.

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