Distribution-independent safety analysis

Problem presented by

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Problem statement

National Air Traffic Services (NATS) are concerned with ensuring low probabilities of errors in determining aircraft positions. In general, error probabilities depend on the tails of some probability distributions for which there has been no theoretical model. Analysis of radar performance is regularly undertaken by NATS to ensure radar performance is within safety limits, with the maximum range being dependent on the declared separation between aircraft. NATS brought two questions to the Study Group, involving the horizontal (azimuthal) errors in radar data and the vertical errors in altimetry system data. In both cases, NATS asked the Study Group to analyse the data and assess whether the probability distributions that are currently used are good models for the errors.

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1 Introduction

NATS brought to the Study Group two problems involving aircraft horizontal and vertical overlap safety margins. In the first problem, azimuthal error data is currently modelled by a double exponential distribution, where the central density is captured by one exponential and the tails by the other. The horizontal overlap probability (the probability that two aircraft are on the same bearing) is determined from tail-tail convolution. We have discovered the azimuthal outliers arise through random errors but also from systematic errors introduced by the way the data is processed. Gibbs phenomena are seen as a result of interpolation error in plotting aircraft paths and we have shown they can be removed by post filtering. In the second problem, vertical error data is currently modelled by a mixed Gaussian exponential distribution, and here we have shown there is insufficient outlier data to fit any distribution, making any estimate of the vertical overlap probability invalid.

2 Problem 1: horizontal safety margins

2.1 Introduction

The National Air Traffic Services are responsible for monitoring and ensuring safe ranges of distance between aircraft. The safety system at NATS requires all radars at its 22 sites to operate at a minimum acceptable level of declared range safety (Target Level of Safety). A typical declaration may take form ‘let radar X support 5Nm separations between pairs of aircraft at any range up to 120Nm from the radar’. Radar performance is inconsistent and varies depending on factors such as icing at the radar, other weather factors, wind farms and modification of the radar itself and its associated equipment. NATS apply the following process to determine the maximum safe range:

- Several hours of azimuthal and range data are recorded from many radars simultaneously.
- The recorded data is post-processed to determine the true position of targets and hence the individual position azimuthal errors θ for each radar return and for each radar (see Figure 1).
- The error data are then analysed statistically by another partly automated process to produce a maximum safe range estimate for a particular radar.

At the last stage, the distribution of azimuthal errors is currently fitted by NATS as a sum of two symmetric exponentials:

$$f(\theta) = \underbrace{\frac{1-a}{2\lambda} e^{-\frac{|\theta|}{\lambda}}}_{\text{Core}} + \underbrace{\frac{a}{2\mu} e^{-\frac{|\theta|}{\mu}}}_{\text{Tail}} \tag{1}$$

for means λ, μ and weighting coefficient $a \ll 1$, ensuring the central part of the density is dominated by the first term and the tails by the second. This mixed exponential distribution is an empirical choice, and there is no theoretical underpinning for the actual

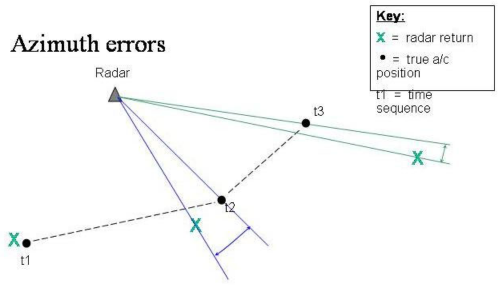


Figure 1: Determining azimuthal error

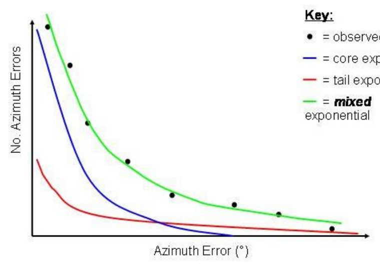


Figure 2: Mixed exponential model

distribution of the data. However, a fitted distribution is required to estimate *Horizontal Overlap Probabilities* (HOPs), that is the probability two aircraft are overlapping when the display in fact shows them have at least the minimum declared separation (*e.g.* 3, 5 or 10Nm). In terms of a convolution, we have

$$HOP = P(|\Theta_1 - \Theta_2| \geq \theta_0) = f * f(\theta_0)\,, \tag{2}$$

where Θ_1 and Θ_2 are independent samples from the distribution. NATS determine HOP from the best fit line of the mixed exponential and consequently can find the maximum safe range. However, the HOP depends on the tail probabilities of the convolution (2), which are poorly understood, and curve fitting, which is very subjective.

2.2 Mixed exponential distribution of data

Here we test the appropriateness of applying a mixed exponential to azimuthal error bin data collected from a radar station at Gatwick Airport. All data was exported to Matlab [1] for analysis, making use of Matlab's graphical packages. Taking the logarithm of the error versus the scaled range, we plotted the blue graph in Figure 4, where we have superimposed in green the logarithm of a mixed exponential distribution. From the approximate straight line in the region $[-0.2, 0.2]$ we see the core is approximately exponentially distributed, however, the tail data looks too random to be modelled by a straight line. The current model assumes tail data is random and identically

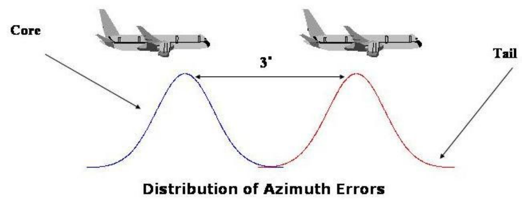


Figure 3: Horizontal overlap convolution

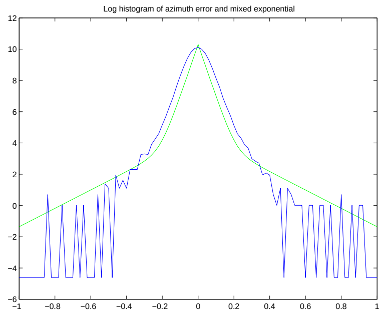


Figure 4: Range and azimuthal data for Gatwick radar

independently distributed (iid). However, upon closer examination of the data this assumption is false.

2.3 Analysis of the azimuthal and range data

Error data was provided for 7 radar stations situated around the UK's main airports and across the UK[1]. As a aircraft comes into range of the radar and is acquired, it is given an id number. Azimuthal and range data was organised as a function of id, which is analagous to time. The rotating radar will pick up the aircraft again on its next rotation, assigning it another id number. Gaps in id number refer to lapses between acquiring a new aircraft or losing track of the current aircraft. Looking at the Gatwick data, we plot in Figure 5 the azimuthal data as a function of id and superimpose range data scaled by maximum range. Data was also manipulated using Matlab to obtain figures 6 and 7. Highlighting a region of ranged data when there is large azimuthal error and zooming in, we notice two types of radar-aircraft interaction: (1) aircraft are picked up by the radar at long range and fly-by resulting in a rectangular hyperbola on the range-id graph; (2) aircraft are suddenly acquired at short range and followed until they are out of range, seen by a 'tick' on Figure 6(a). Looking at Figure 6(b) it is clear that random error coincides with the start of the acquisition of an aircraft path, resulting in highest azimuthal error.

[1]The 7 stations were Burrington, Claxby, Gatwick, Heathrow10, Heathrow23, Jersey and Mount Gabriel.

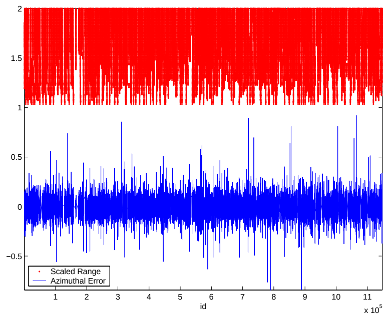
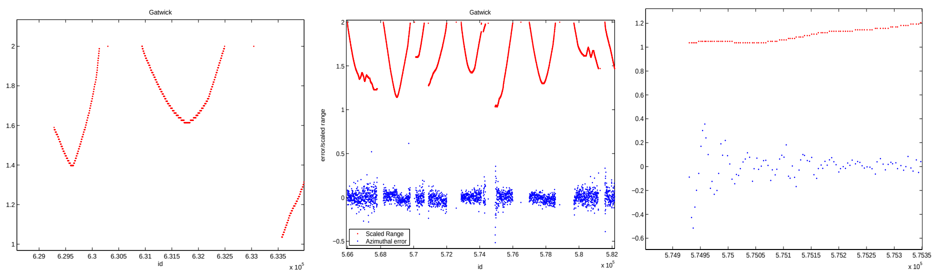


Figure 5: Log histogram of Gatwick data



(a) Typical radar-aircraft interaction

(b) Zoom on region of large error

(c) Acquired aircraft azimuthal error

Figure 6: Tracking azimuthal errors

The width of the hyperbolae seen in Figure 6 is determined by the distance of the aircraft from the radar. Consider a fly-by, where the range from aircraft to radar is denoted by r, the speed of the aircraft by v, and L is the minimum distance to the radar which we scale to time $t = 0$. Then the range is given by the hyperbolic equation $r^2 = L^2 + v^2t^2$.

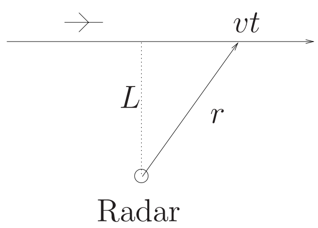


Figure 7: Calculating range on a fly-by

Now by zooming into the azimuthal error when an aircraft is acquired, shown in Figure 6(c), we see oscillatory errors resembling Gibbs phenomena. This is often related to interpolating square functions with polynomials (for example, see Burden and Faires [2], Chapter 3.4). Hence applying Fourier analysis, we take the fast Fourier transform (FFT) of the azimuth error. The FFT plotted against id is shown in Figure 8. It is very apparent from the FFT there is a signal peak in this data corresponding to systematic

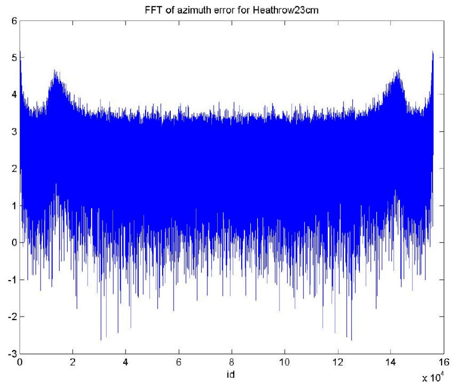


Figure 8: Fourier transform of Gatwick error data

oscillatory error with very marked peaks, quite different from what one would expect from random errors. Given that there is a total of 157,000 data points (N), we can determine the frequency of oscillation from the location of a peak. Looking at the peak occurring at 13,000, the frequency of oscillation is such that

$$f_{\text{peak}} = \frac{id_{\text{peak}}}{N} = \frac{13000}{157000} \approx \frac{1}{12}\,. \tag{3}$$

Hence, there are 12 points per wave from this systematic error (see Figure 6(c)). The same Gibbs phenomenon is seen in the data for the other radar stations. We now investigate the source of error, by looking at the way radars acquire aircraft.

2.4 Systematic source of error

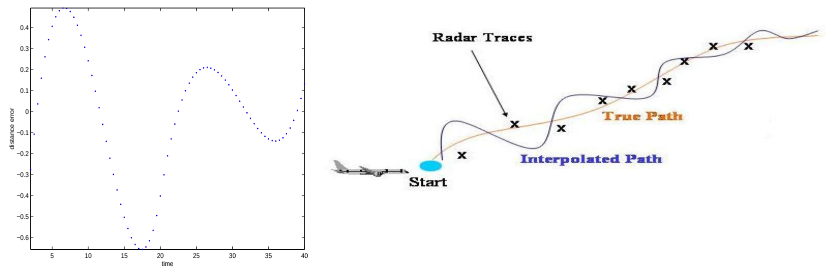
Azimuth and range data were provided for each radar station. In general most radars tracked aircraft flying at close range ($<$ 180Nm), but a few tracked aircraft at both close and far range ($>$ 1,000Nm). We hypothesise that on take-off, the aircraft is tracked by an airport radar station capturing the aircraft on every rotation. The path of the aircraft must then be interpolated, leading to large azimuthal errors from close-range observation. However, the farther the aircraft is from the airport the more likely it will be picked up by another radar from long range, providing more radar traces and a better interpolated path.

Let us consider a hypothetical path of an aircraft described by $f = 400(1-e^{-t\ln(2)/20})$. Radar readings are taken every Δt seconds and we perform a cubic spline to approximate the path. After 20 seconds, the aircraft is picked up by a second radar and we approximate the path again with a cubic spline.

Hence, acquiring aircraft introduces interpolation errors which become recorded as false azimuth errors in the data.

2.5 Removal of systematic error

We apply a band-gap filter to remove the component of frequency 1/12 in the FFT data (also seen in the Heathrow radar). An inverse FFT is then applied. In Figure 10 we

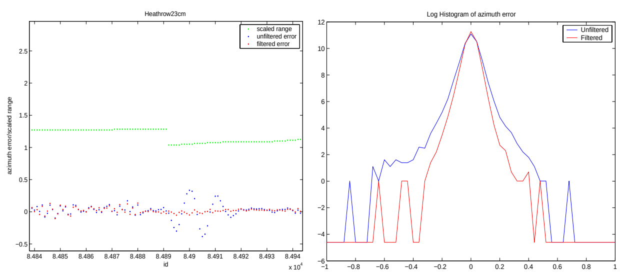


(a) Simulated spline error

(b) Aircraft path tracked by radar

Figure 9: Simulated spline error in acquiring an aircraft path by radar

plot our original noisy azimuthal data (in blue) with the new filtered data (in red). We



(a) Filtered and noisy azimuthal error

(b) Log histogram of filtered error

Figure 10: Filtered errors

have clearly removed the systematic error in Figure 10(a) and taking the log histogram of the azimuthal error (Figure 10(b)) shows a dramatic change in the tail distribution. The clearer data appears to have a core normal distribution, and in Figure 11 we test the sample data against a standard normal using a quantile-quantile plot. In fact, the core data reasonably fits a normal distribution, but the tail data does not.

2.6 Conclusion and recommendations

Analysis of the azimuthal error shows that systematic error is responsible. Errors in interpolating aircraft paths lead to large errors in azimuth data. Applying an FFT, filtering out these errors, and using an inverse FFT yields clearer data, reducing the magnitude of the largest errors. At present NATS are using noisy data to calculate the HOP, and are producing safety margins that are larger than necessary.

Whilst we can remove the error we could also lose information, so we recommend to NATS that in the calculation of aircraft paths they remove the systematic error by using

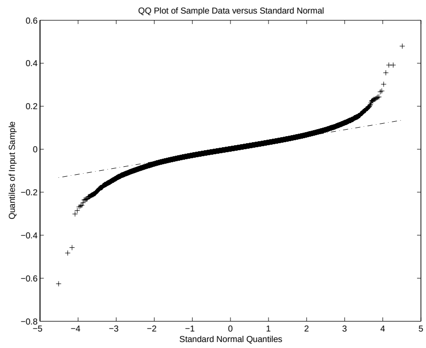
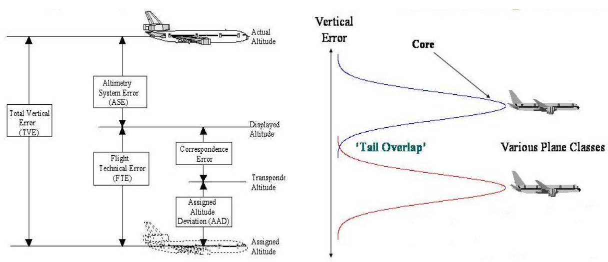


Figure 11: Quantile-quantile plot

an alternative to interpolation, such as the use of GPS data. Having removed the error, maximum likelihood estimation should be used to fit data to the distribution.

3 Problem 2: vertical safety margins

In problem 1, we dealt with azimuthal error, an angular error in determining the position of an aircraft. However, NATS are also interested in *total vertical error*, that is the difference in height between an aircraft's assigned altitude and its actual altitude. An aircraft determines its altitude from its altimeter which works out height from outside pressure. In bad weather conditions an altimeter may display an inaccurate altitude. The difference between actual altitude and displayed altitude is known as *altimetry system error*.



(a) Height-keeping performance

(b) Vertical overlap convolution

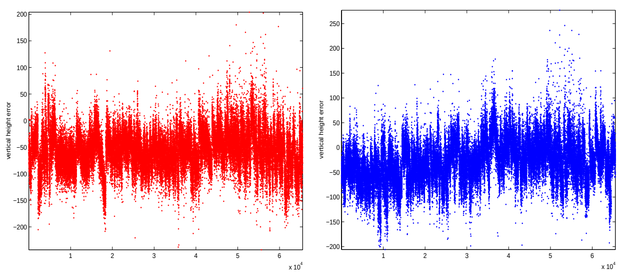
Figure 12: Vertical height error and overlap

Total vertical error data has been collected from two Height Monitoring Units (HMUs) located near Strumble and Gander. From this set of data, the altimetry system error has been calculated and was provided to the Study Group, to be fitted with a pdf. Determining the distribution will allow NATS to calculate the vertical overlap probability P_z. According to the so-called Reich Model, the collision risk N_{az} is a function of vertical overlap probability, horizontal overlap frequency N_r and a weighting kinematic factor K such that $N_{az} = P_z \times N_r \times K$. The height data is believed to be distributed by a *Mixed Gaussian Exponential* modelled by

$$f(d) = \underbrace{(1-\alpha)\left(\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{d^2}{2\sigma^2}\right)\right)}_{\text{Gaussian}} + \underbrace{\alpha\left(\frac{1}{2\lambda}\exp\left(-\frac{|d|}{\lambda}\right)\right)}_{\textit{Exponential}} \tag{4}$$

for weighting parameter $\alpha \ll 1$ so that the core is dominated by the Gaussian distribution. The Study Group was asked to fit the data to this distribution and find the probability of a tail overlap. The data was supplied as 103 data sets, where each set i has N_i observations of height errors f_i for aircraft type i.

Consider two examples: the Boeing 737-400 series and the Boeing 737-500 series, for which data is shown in Figure 13. The Boeing 400 series has mean error -56.7 and



(a) Boeing 737-400 Series

(b) Boeing 737-500 Series

Figure 13: Boeing 737 ASE data

standard deviation 36.2, and the 500 Series has mean -28.0 with standard deviation 41.2. Observing the data, there is little consistency. Other data sets have positive mean and smaller standard deviation. Unfortunately the data was given as pure height error alone for a series of aircraft. A trend could appear if the height error had a time stamp and was given for a particular aircraft. In Figure 14 we consider the aggregate data and plot a histogram with log histogram superimposed.

Again we see inconsistent outliers which would make fitting a pdf very difficult. To determine whether the data fits a mixed Gaussian exponential distribution, we calculate the expectation, defined by

$$E(d) = E(X-d|X>d) = \frac{\int_u^\infty (s-u) f_x(s)ds}{\int_u^\infty f_x(s)ds}. \tag{5}$$

The expectation of the Gaussian core is given by $E(d) = \sigma^2/d$, a hyperbola in d, and an exponential tail expectation $E(d) = 1/\lambda$ would give a straight line as a function of d. In Figure 15 the expectation of the aggregate data is plotted.

The core appears to be Gaussian, but the tail data is far from constant. It is too random to fit and should not be used to calculate probability overlap.

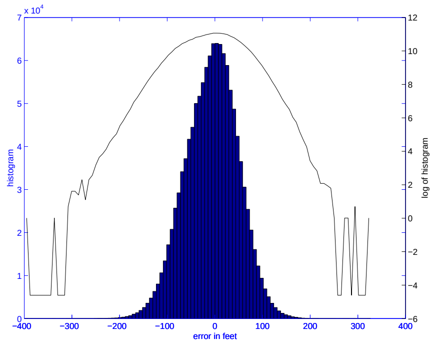


Figure 14: Aggregate data log histogram and histogram plot

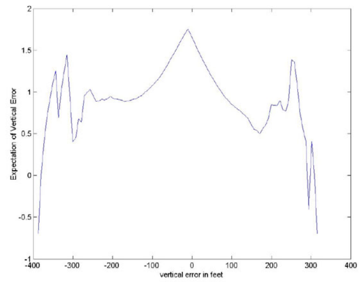


Figure 15: Expectation of aggregate data

3.1 Conclusions and recommendations

NATS use a mixed Gaussian-exponential model of ASE data to calculate the vertical probability overlap of aircrafts. However, analysis of the data suggests that more data is required for the outliers. There is currently not enough tail data to fit conclusively to any pdf. The data would be of better use if it were time stamped, since it would then elucidate whether the altimeters are being maintained to compensate for ASE; if so, this would suggest the data is no long iid and cannot be fitted to a pdf.

References

[1] D.J. Higham, N.J. Higham *Matlab Guide* (SIAM, Philadelphia, 2000)

[2] R.L. Burden, J. Douglas Faires. *Numerical Analysis* (Brooks/Cole Publishing Company, USA, 1997)