THERMAL FOOD PROBE

MAFF, CSL-Food Science Laboratory, Torry.

1. INTRODUCTION

The objective of the work on this study group problem was to aid in the development of a thermal probe to sense the chemical composition of foods from measurements of their thermal properties, in particular to determine the water, protein and fat composition of a fish from estimates of its thermal conductivity k, diffusivity κ and heat capacity c.

The basic geometry of the existing thermal probe comprises a circular heater of radius a which is heated electrically. When placed on the skin of the fish, the heat flux from the probe causes a temperature rise $T(\mathbf{x}, t)$ in the fish, and the temperature at the heater surface is measured by monitoring changes in the resistance of the heating element.

We illustrate this geometry below.

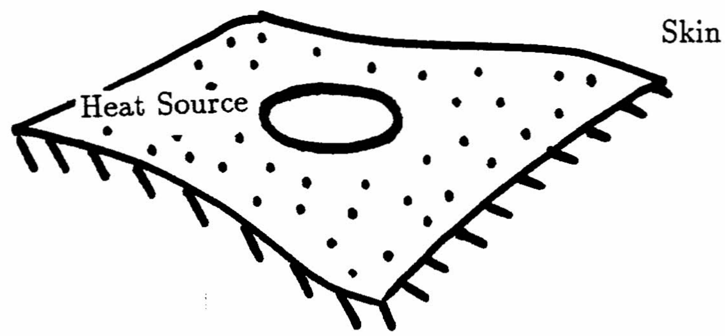

Figure 1. The surface of the fish with the heater.

An actual fish probably has an inhomogeneous, anisotropic thermal diffusivity which will be affected by differing properties of skin and muscle layers. In addition, blood vessels may provide paths along which heat is rapidly convected. It is very difficult to assess all of these possibilities and so to make progress it was assumed that, within the body of the fish, the medium was uniform, and heat transfer purely conductive being governed by the usual equation

$$T_t = \kappa \nabla^2 T \tag{1}$$

At the surface of the heater there is a constant flux of heat given by

$$Q = -k \frac{\partial T}{\partial n}, \tag{2}$$

where $\mathbf{n}$ is the normal into the fish from the heater surface. It was assumed that elsewhere on the skin of the fish the air acted as a thermal insulator so that

$$0 = -k \frac{\partial T}{\partial n}. \tag{3}$$

The objectives of the study were then as follows

- To solve (1)(2)(3) for the illustrated geometry to determine $T(\mathbf{x}, t)$ at various points within the fish.
- To determine how accurately k, κ, c could be estimated from measurements of T on realistic time-scales.
- To determine how accurately the water/fat/protein content could then be estimated.
- To advise on other possible probe designs.

One clear conclusion from this study is that the accuracy of the probe is significantly limited by the time scales involved and also that, whilst the water composition can be calculated accurately, it is far harder to determine the relative fat/protein composition.

In addition to the thermal probe, other detection procedures were considered including sensing the odour of the fish!

2. SOME CONSTANTS AND PRELIMINARY ESTIMATES

The thermal diffusivity κ of a "typical" fish is around

$$\kappa \sim 10^{-7}\ m^2 s^{-1}, \tag{4}$$

which imposes a severe restriction on the applications of the device as heat will only penetrate a small distance of around 1mm into the fish on the timescale of 10s or so which would be reasonable for an experiment. Thus the probe can only give very local estimates of the thermal properties of the fish. One approximation that can be made (based opon this estimate) is to treat <u>all</u> fish (flat or otherwise) as infinite half-planes extending in the direction $z \to \infty$ with the heater on the surface at $z = 0$ as the thermal wave from the probe will not reach the extremities of the fish in the experimental time-scale.

3. A ONE-DIMENSIONAL SOLUTION

The (hot disk, TPS) thermal probe, as currently manufactured, has a radius a where

$$a \sim 1mm - 1cm. \tag{5}$$

A typical time-scale for the thermal wave to travel over the length scale defined by the probe is then given by

$$\tau = \frac{a^2}{\kappa} \sim 10s - 1000s. \tag{6}$$

If $t << \tau$ then the probe can be approximated as an infinite heat source and hence for a preliminary calculation we consider a one-dimensional problem with the fish uniformly heated at $z = 0$. This gives

$$T_t = k\ T_{zz}, \quad -kT_z = Q \quad \text{at} \quad z = 0 \tag{7}$$

This can be solved easily (see Carslaw & Jaeger) to give

$$T(z,t) = \frac{2Q\sqrt{\kappa t}}{k} \text{ ierfc } \left(\frac{|z|}{2\sqrt{\kappa t}}\right) \tag{8}$$

Thus, the temperature measured on the surface is given by

$$T(0,t) = \frac{2Q}{\sqrt{\pi}} \alpha t^{\frac{1}{2}} \tag{9}$$

where

$$\alpha = \sqrt{\kappa}/k. \tag{10}$$

For a given Q, the function α can then be determined from a plot of T against $t^{\frac{1}{2}}$, but separate estimates of κ and k cannot be made.

A calculation based upon data given on the values of k and κ shows that α takes similar values for fat & protein but is different for water. Thus this simple calculation shows that on small time-scales a useful estimate can be obtained for the water content of the fish.

4. A FULLY THREE-DIMENSIONAL SOLUTION

To determine κ and k separately we must introduce a finite dimensional geometry into the problem considered in Section 3. Mathematically, the easiest case to consider is a spherical heat source in an infinite fish. This satisfies

$$T_t = \kappa(T_{rr} + \frac{2}{r} T_r), \qquad -kT_r = Q \text{ at } r = a \tag{11}$$

where r is the distance from the centre of the probe.

The solution of (1) at $r = a$ is given by

$$T(a,t) = \frac{Qa}{k}\left[1 - e^{kt/a^2} \text{ erfc } \frac{\sqrt{\kappa t}}{a}\right] \tag{12}$$

If $t/\tau \equiv \frac{kt}{a^2}$ is small then

$$T(a,t) = \frac{2}{\sqrt{\pi}} Q \alpha t^{\frac{1}{2}} - \frac{Q\kappa}{k} t + O\left(\left(\frac{t}{\tau}\right)^{\frac{3}{2}}\right) \tag{13}$$

which we see comprises the one-dimensional solution (9) as a leading order term plus an additional term due to the three-dimensional effects. The magnitude of this term could, in principle, be determined even when t is still small compared with τ. This would in turn give an estimate for κ/k. Combined with the leading order estimate for α this will give κ and k independently over a time scale which is not strictly limited by the estimate (6).

If t/τ is large then

$$T(a,t) \to \frac{Qa}{k} \equiv T_\infty \tag{14}$$

and an estimate of T_∞ (over this longer time scale) will give k accurately.

(Graphs of $T(t,a)$ and $T(t,a)/t^{\frac{1}{2}}$ are given in Figs 2 and 3 for $a = k = \kappa = 1$).

It is interesting to note that k, κ and indeed T_∞ can be estimated accurately from $T(t,a)$ by using a rational approximant P. To do this a few measurements of $T(t,a)$ should be made at times t_i (which need not be large compared to τ) and a rational interpolant made to $T(t_i, a)$. For example estimating coefficients $b, c, d \ldots, h$ such that

$$T(a,t_i) = P(t_i) \equiv \frac{b\, t_i^{\frac{1}{2}} + c\, t_i + d\, t_i^{\frac{3}{2}}}{e + f\, t_i^{\frac{1}{2}} + g\, t_i + h\, t_i^{\frac{3}{2}}} \tag{15}$$

This can be done robustly in a standard numerical procedure. The value of T_∞ can then be estimated from

$$T_\infty \approx \frac{d}{h}, \tag{16}$$

and hence a value of k can be estimated. To demonstrate the effectiveness of this procedure we calculated the [3,3] Padé rational approximation to the function $f(s)$ derived from T when $a = k = \kappa = 1$ so that

$$f(s) = 1 - e^{s^2}\ \mathrm{erfc}(s) \equiv T(a, s^2) \tag{17}$$

(The Padé approximation to $f(s)$ matches its Taylor series at $s = 0$ and hence uses information about $f(s)$ close to $s = 0$ which would correspond to making measurements on small time-scales).

The estimate of $f(\infty)$ from (16) was then 1.01978 which compares very well with the time value of 1 (even though the former estimate is only based upon information at $s = 0$). (The Padé approximation is given in the following Maple calculation and compared with $f(s)$ in Fig 4.)

```
> f:=1-exp(s^2)*erfc(s);
```

$$f := 1 - e^{s^2}\,\mathrm{erfc}(s)$$

```
> g:=pade(f,s,[3,3]);
```

$$g := \frac{1}{30}\Big((-307200 + 97200\,\pi)\, s + \left(67200\sqrt{\pi} - 21600\,\pi^{3/2}\right) s^2 + \left(-81920 + 68400\,\pi - 13500\,\pi^2\right) s^3 \Big) \Big/ \Big(-5120\sqrt{\pi} + 1620\,\pi^{3/2} + \left(-1440\,\pi + 450\,\pi^2\right) s + \left(-660\,\pi^{3/2} + 2048\sqrt{\pi}\right) s^2 + \left(704\,\pi - 225\,\pi^2\right) s^3 \Big)$$

```
> evalf(limit(g,s=infinity));
```

$$1.019781455$$

We see from this discussion that the spherical probe in an infinite fish can give good estimates for κ, k relatively quickly which can also be computed reliably.

Of course this geometry may be difficult to realise in practice!

5. THE DISC HEATER

We now consider the original geometry of a disc heater on a semi-infinite fish. In this case we have

$$\left\{\begin{array}{lcll} T_t &=& \kappa\left(T_{rr} + \frac{1}{r}\, T_r + T_{zz}\right) & z > 0, \\ Q &=& -k\, T_r & z = 0,\ r < a \\ 0 &=& -k\, T_r & z = 0,\ r > a \end{array}\right\} \tag{18}$$

A solution of this problem may also be found in Carslaw and Jaeger, chapter 8, expressed as an infinite sum of Bessel functions. However, if $r = a$ the expression simplifies to give

$$T(r=0,z,t) = 2Q\alpha t^{\frac{1}{2}}\left[\operatorname{ierfc}(z/2\sqrt{\kappa t}) - \operatorname{ierfc}\left((z^2+a^2)^{\frac{1}{2}}/2\sqrt{\kappa t}\right)\right] \tag{19}$$

so that if a measurement is made at $z = 0$ (on the surface of the disc) we have

$$T(0,0,t) = 2Q\alpha t^{\frac{1}{2}}\left[\frac{1}{\sqrt{\pi}} - \operatorname{ierfc}\left(\frac{a}{2\sqrt{\kappa t}}\right)\right] \tag{20}$$

If $(t/\tau) \ll 1$ this gives

$$T(0,0,t) \sim \frac{2Q\alpha}{\sqrt{\pi}}\, t^{\frac{1}{2}} + \mathcal{O}(e^{-(\tau/t)^{\frac{1}{2}}}). \tag{21}$$

Unlike the related expression (13) for the spherical probe we see that the "three- dimensional effects" in (21) are exponentially small when compared to the leading order "one-dimensional" term. Thus for this geometry it is impossible to determine κ and k independently until $t = \mathcal{O}(\tau)$.

For large t, $T \to T_\infty$ as in (14) and hence k can then be estimated. The functions $T(0,0,t)$ and $T(0,0,t)/t^{\frac{1}{2}}$ are plotted in Figs 5,6.

Alternatively, a measurement could (in principle) be made at a point $z \neq 0$. In this case, T is exponentially small until a time $t(z)$ where

$$t(z) = \mathcal{O}(z^2/4\kappa) \tag{22}$$

Thus by measuring $t(z)$ ($\&T(0,z,t)$) we may estimate κ directly on a smaller time-scale than τ (provided that $z < a$).

Plots of $T(0, z=2, t)$ are given in Figs 7 and 8.

Note In practice measurements of T for the disc heater are made at $z = 0$ and take an average over the disc surface. Obtaining a closed form expression for this case appears to be hard!

6. OTHER PROBE GEOMETRIES

We conclude that the spherical probe gives good results on small time-scales but is hard to implement, whereas the disc (with measurements only at $z = 0$) will only give information over time scales which are long.

Most other probes suffer from similar problems. An example is a needle probe of radius a with T measured on the probe surface i.e.

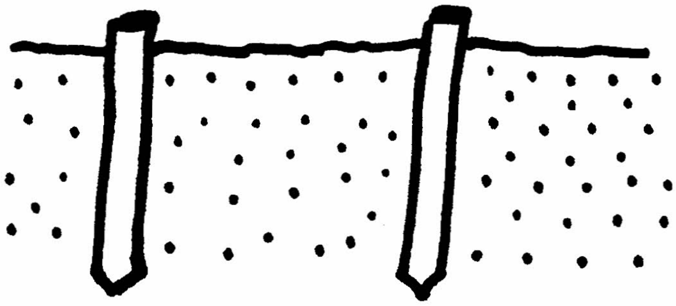

Figure 9. A needle probe geometry.

In this case

$$T \sim \frac{Q}{4\pi k} \log(t) \text{ if } t \gg \frac{a^2}{\kappa} \tag{23}$$

so that k can be estimated. If two such probes are inserted into the fish and one is heated, then k can be estimated from the time it takes the thermal wave to go from one to the other.

7. CALCULATION OF THE WATER, FAT AND PROTEIN CONTENT

Having determined k and κ to (hopefully) a reasonable degree of accuracy - - and hence having determined the specific heat capacity c of the fish – it must then be determined how accurately the water, fat and protein composition can be calculated.

Empirical studies with different food stuffs have suggested a linear model such that if A_w, A_f and A_p are the water, fat and protein compositions then

$$M \begin{pmatrix} A_w \\ A_f \\ A_p \end{pmatrix} = \begin{pmatrix} 1 \\ c/4180 \\ k \end{pmatrix} \tag{24}$$

where

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0.5 & 0.3 \\ 0.56 & 0.09 & 0.16 \end{pmatrix} \tag{25}$$

The matrix M has a condition number of 17.49 indicating that errors in c and k will be amplified in calculating A_f, A_w, A_p. (Although it should be pointed out that the first equation in (24) namely that

$$A_f + A_w + A_p = 1$$

will always be satisfied identically and hence the actual error in the solution will be smaller than those implied by the condition number.) A complete error analysis can be determined through a Singular Value Decomposition (SVD) of M. These results may most easily be expressed geometrically as follows: the values of A_w, A_f, A_p are constrained to lie in the triangle $\triangle_1$ and are mapped to points in a similar $\triangle_2$ with identification of the vertices a, b and c. It is significant that the equilateral triangle $\triangle_1$ is mapped to an isosceles triangle $\triangle_2$ with aspect ratio 5 : 1 (see the figure below).

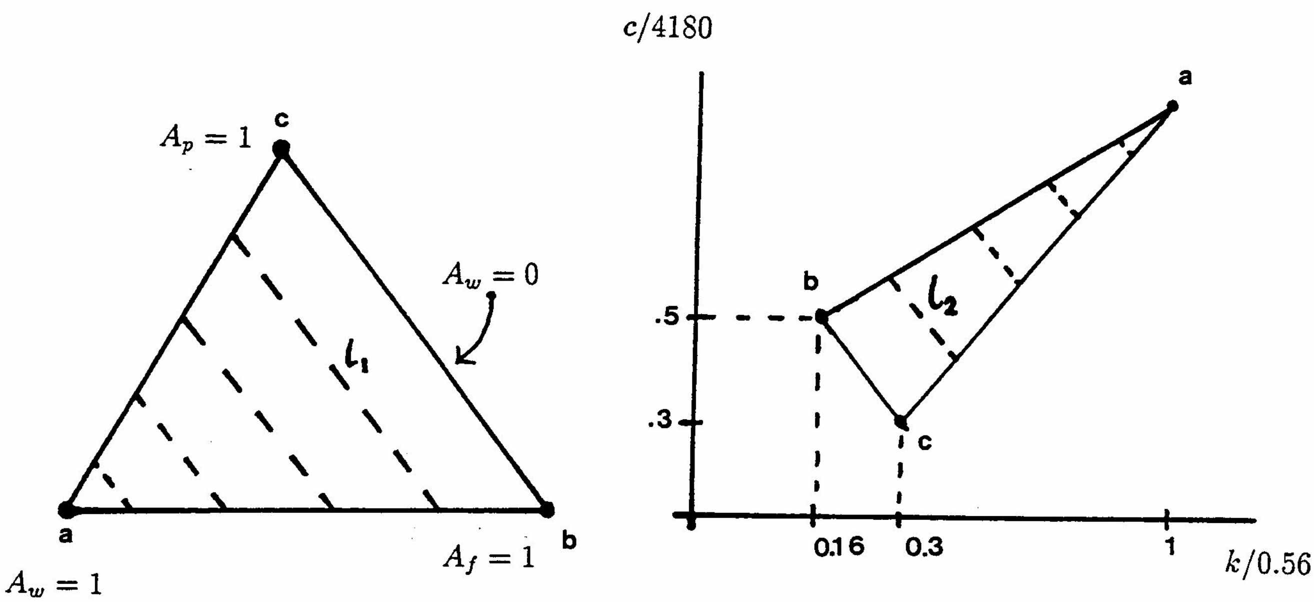

Figure 10. The relationship between A_w, A_f, A_p and c, k.

Indicated on both triangles are lines along which A_w is constant. If ℓ_2 is such a line on $\triangle_2$ and is in turn mapped by the operator M^{-1} onto the line ℓ_1, then a small error in c, k in a direction parallel to ℓ_2 is multiplied by 5 when it is mapped to ℓ_1.

In other words: small errors in the estimates of k and of c are amplified into larger errors in the individual estimates of A_f and A_p. Similarly, errors in k and c orthogonal to ℓ_2 are reduced under the action of M^{-1} and hence the value of A_w may always be estimated accurately!

It should be pointed out that the evidence for the (linear) model (24) is limited. It was suggested by Professor Fornberg (Exxon Research) that a more realistic representation could be obtained by fitting experimental data with a radial basis function approximation to give a relation between k and c to A_w, A_f and A_p..

It is clear from the above calculations, that thermal measurements are never going to be good in determining the relative fat/protein content of the fish, and that some other measurement will be necessary – for example a measure of the bouyancy of the fish to give the fat content. Another possible such measure would be of the smell of the fish, as this would give an indication of its chemical composition. There was a brief discussion about this possibility but too little information was available to establish its viability.

8. CONCLUSIONS

There are two clear conclusions from this work

(1) A disc-type probe will only give information about thermal properties over relatively long time-scales which may rule out its practicality.

(2) Thermal properties (with associated errors) will give accurate estimates of the water content of the fish but not of its relative fat/protein content.

The result in (1) can be improved by using a better design of the probe but the limitations in (2) imply the need for other than thermal measures.

CJB, HH-O, NF, JL, GW, DSR

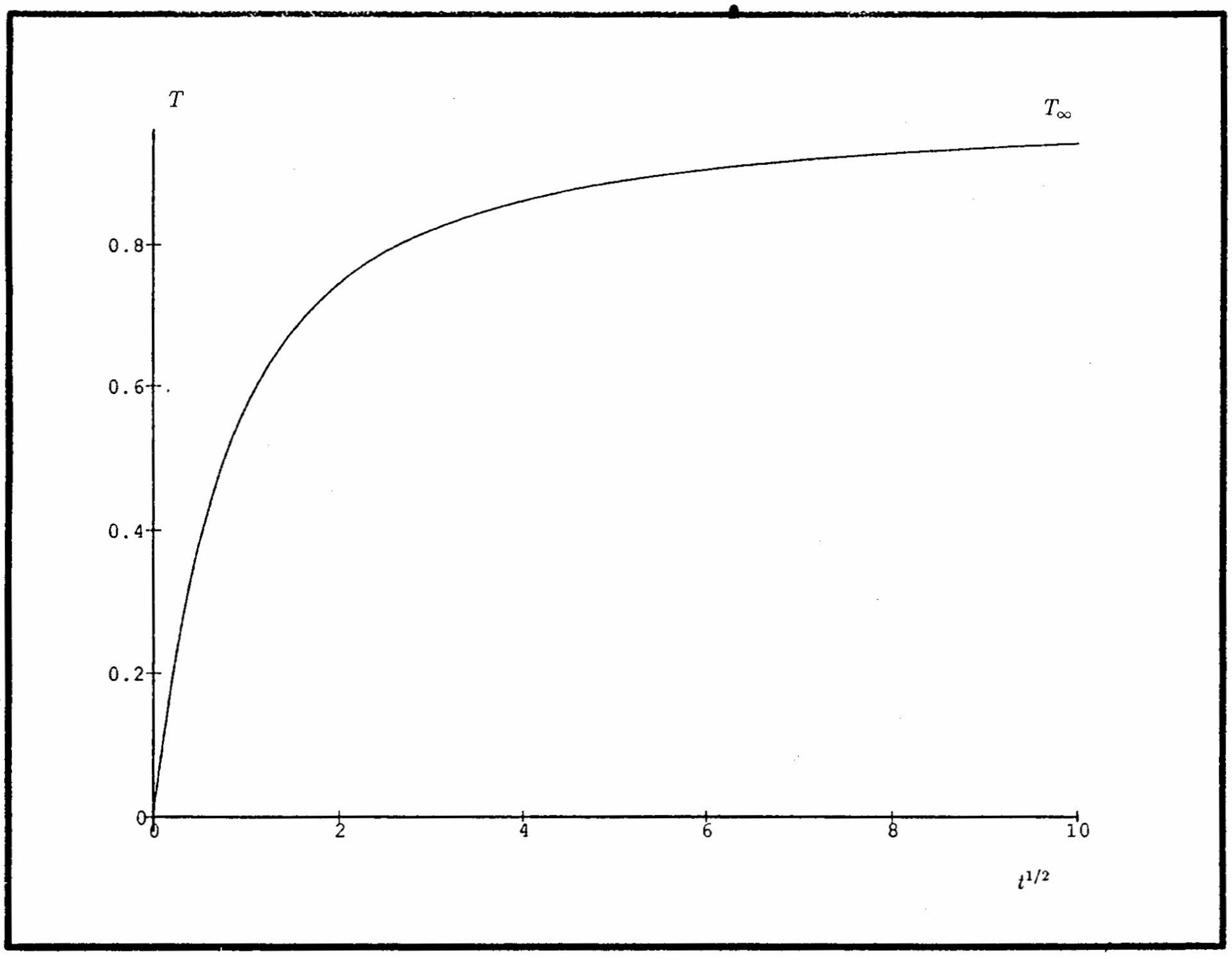

Figure 2. The rise in the temperature of the spherical probe over long times as a function of $t^{1/2}$.

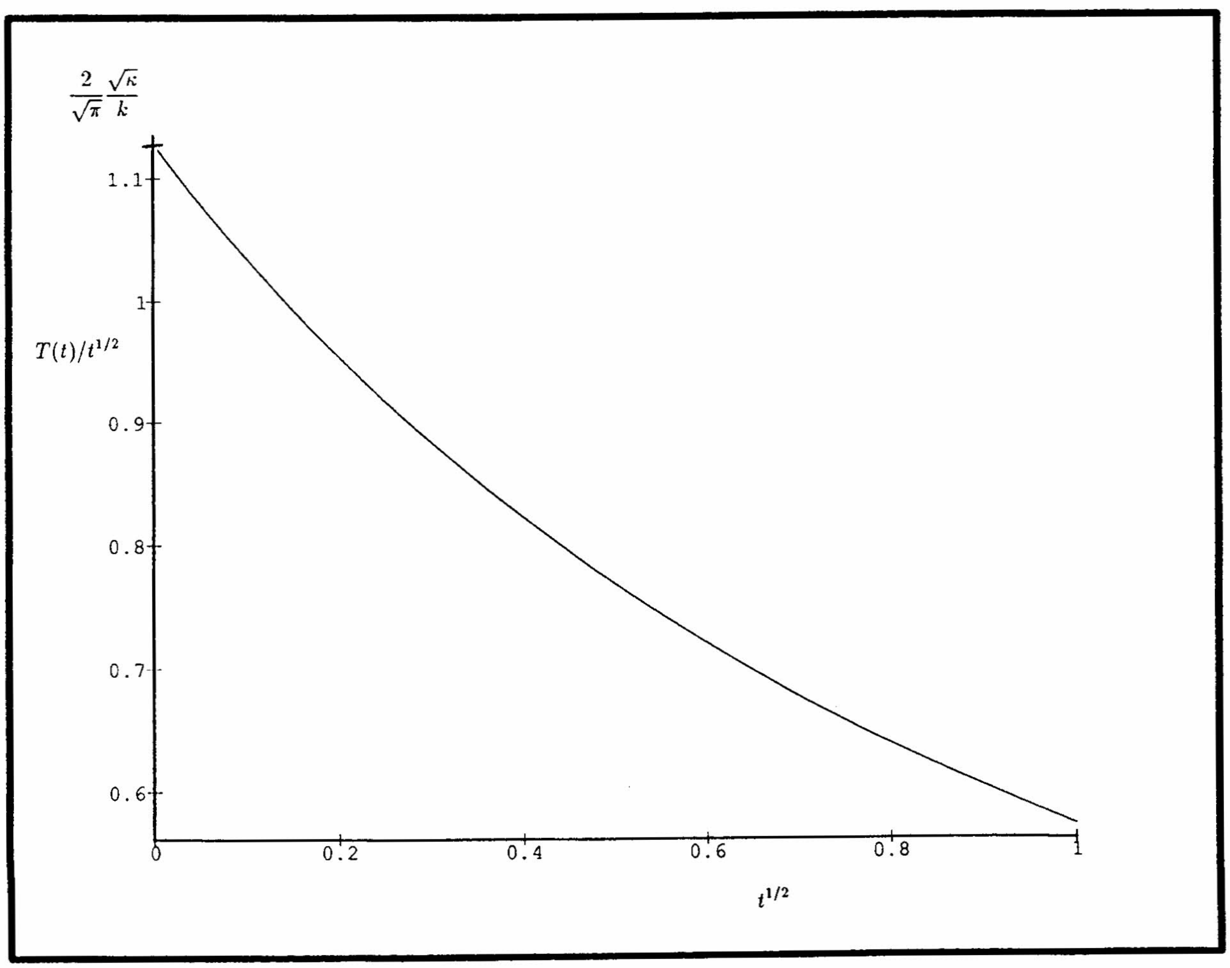

Figure 3. The rise in the temperature of the spherical probe over short times showing $T/t^{1/2}$ as a function of $t^{1/2}$.

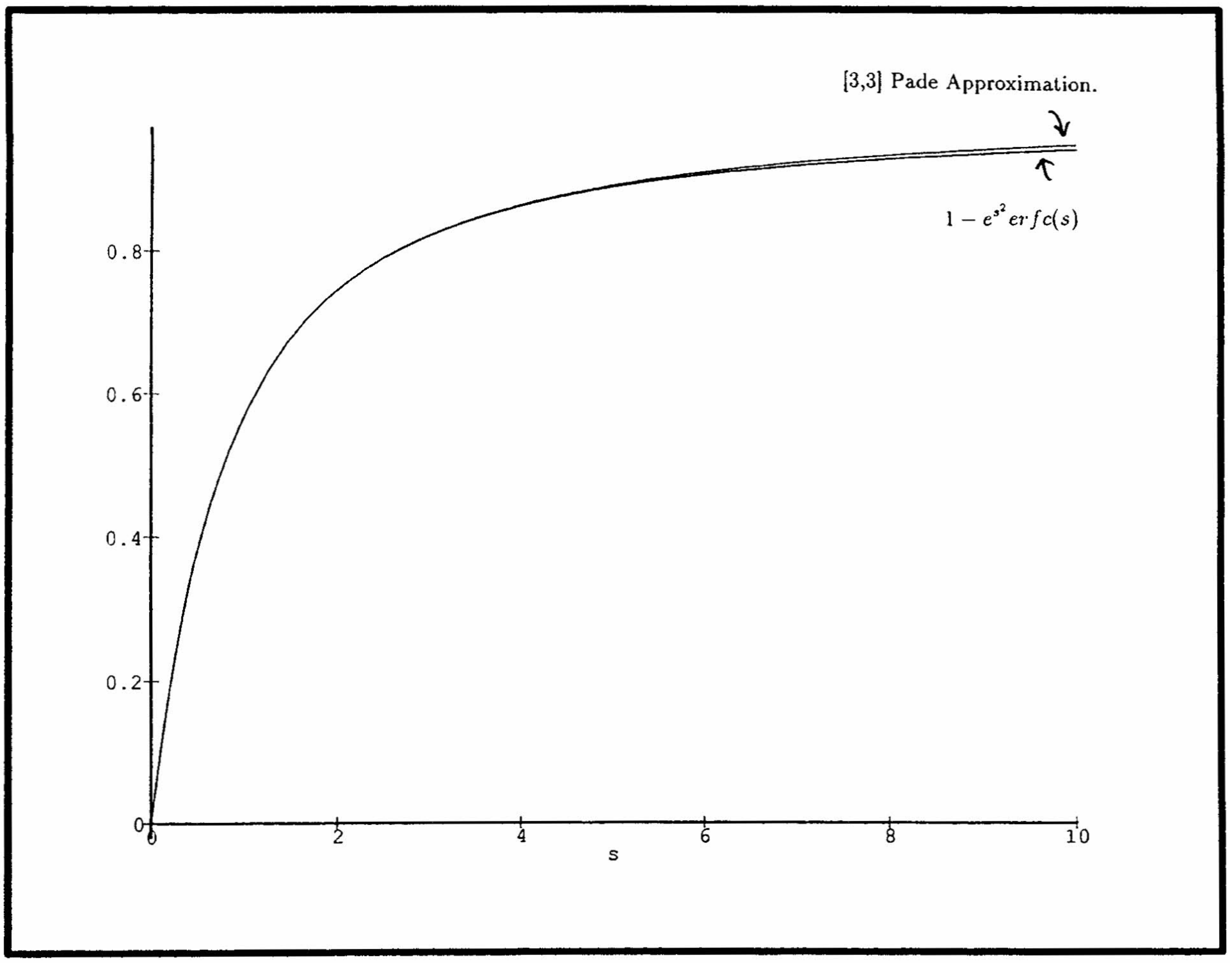

Figure 4. A comparison between the temperature profile and its Pade approximant.

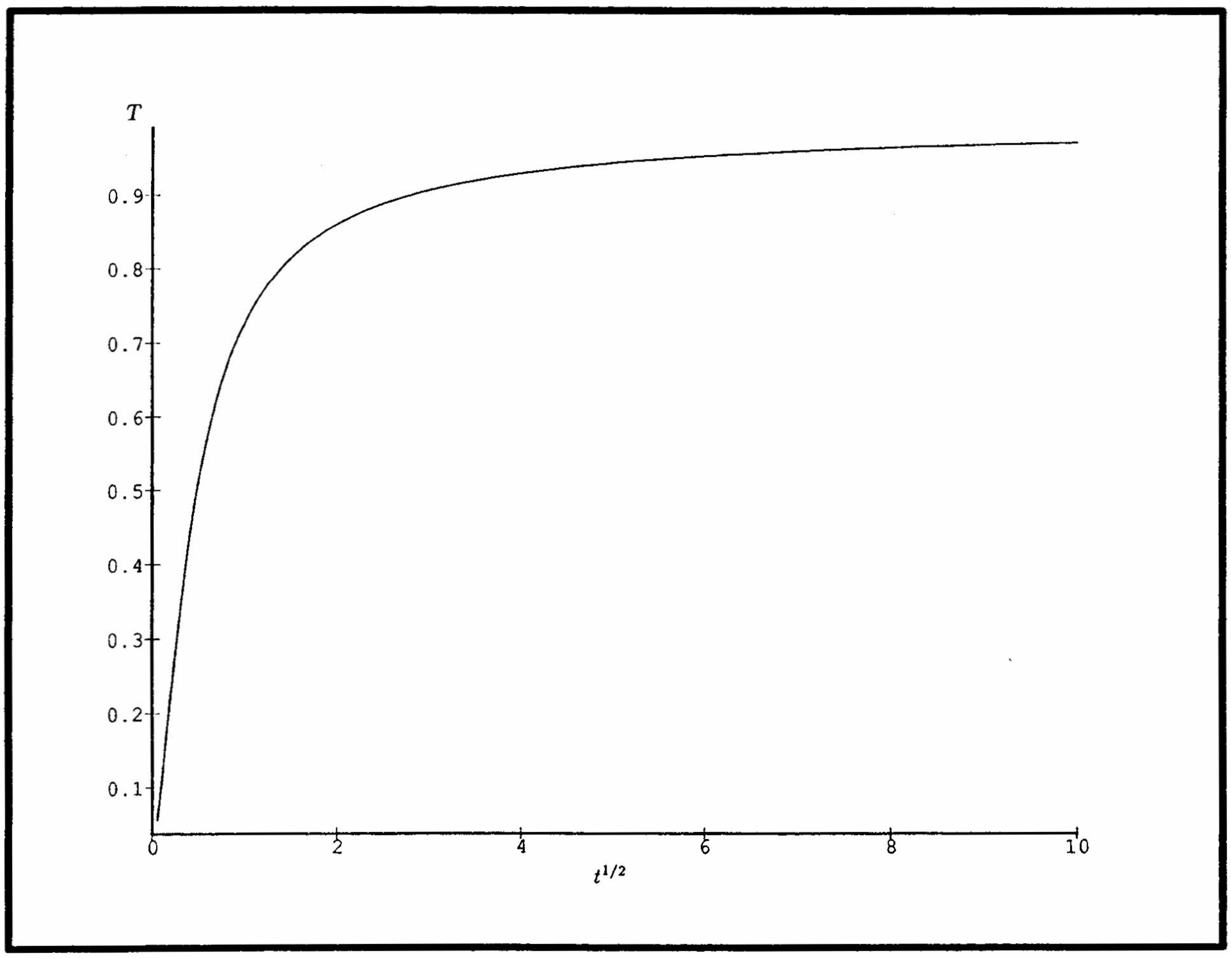

Figure 5. The rise in the temperature of the disc probe on the surface of the fish over long times.

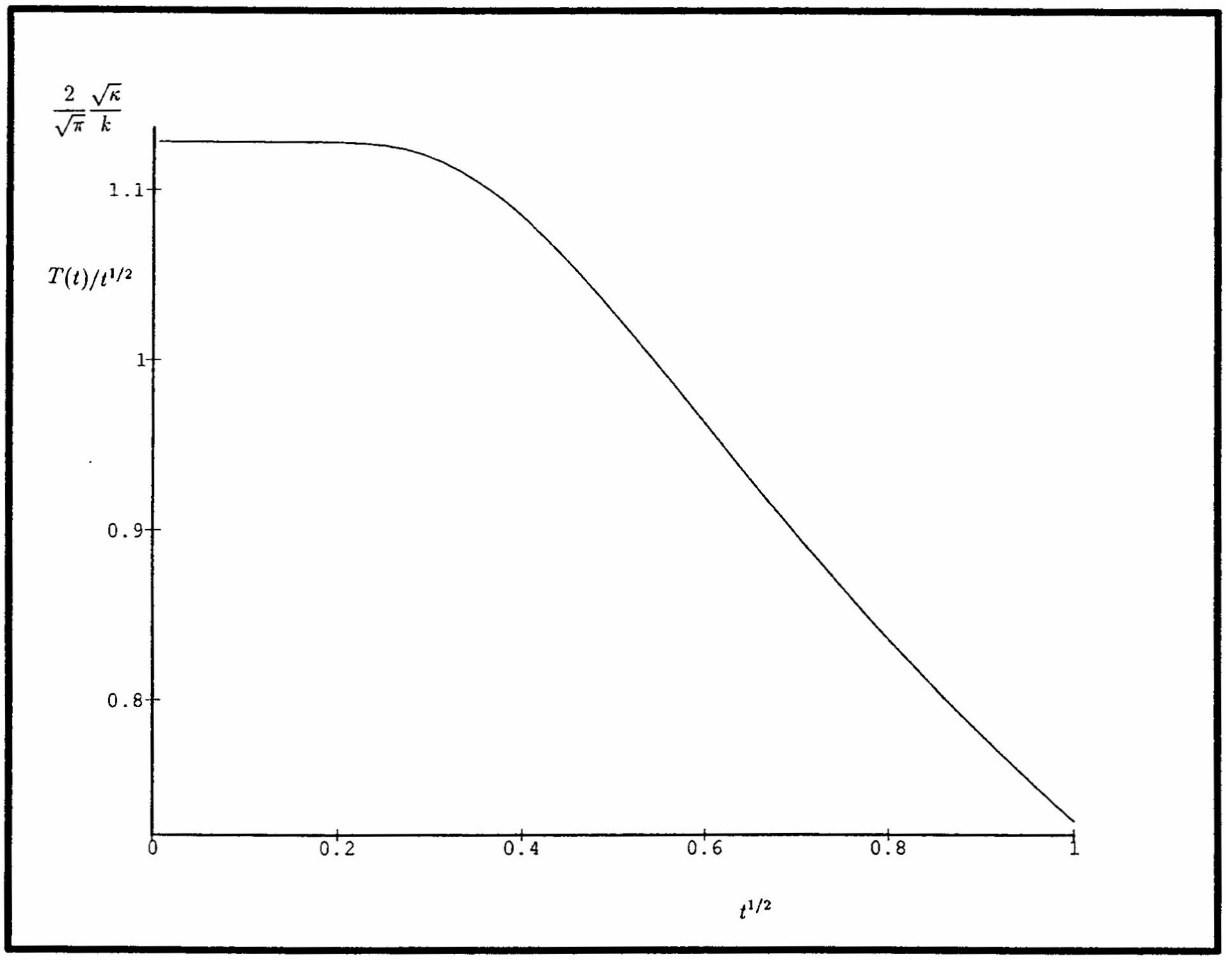

Figure 6. The rise in the temperature of the disc probe on the surface of the fish over short times.

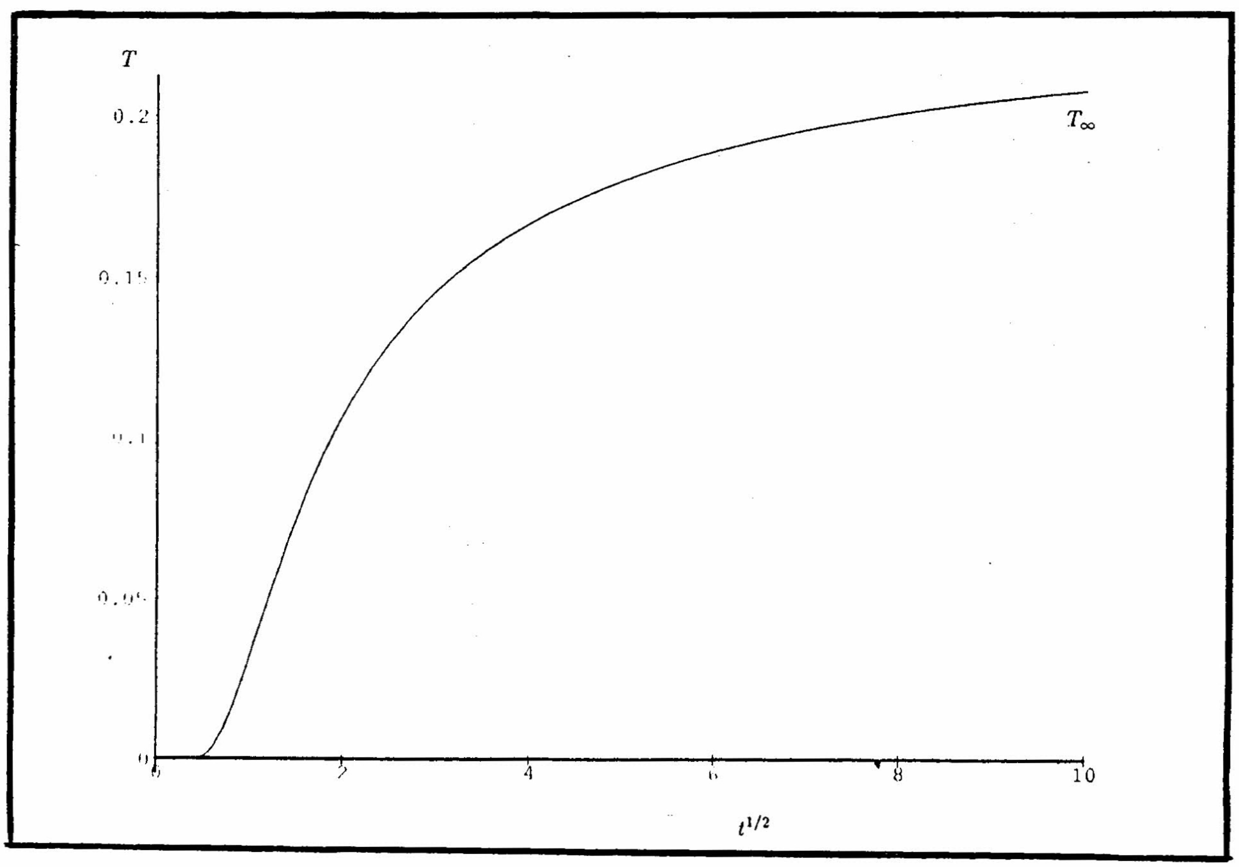

Figure 7. The rise in the temperature of the fish at a distance of 2 below the disc probe over long times.

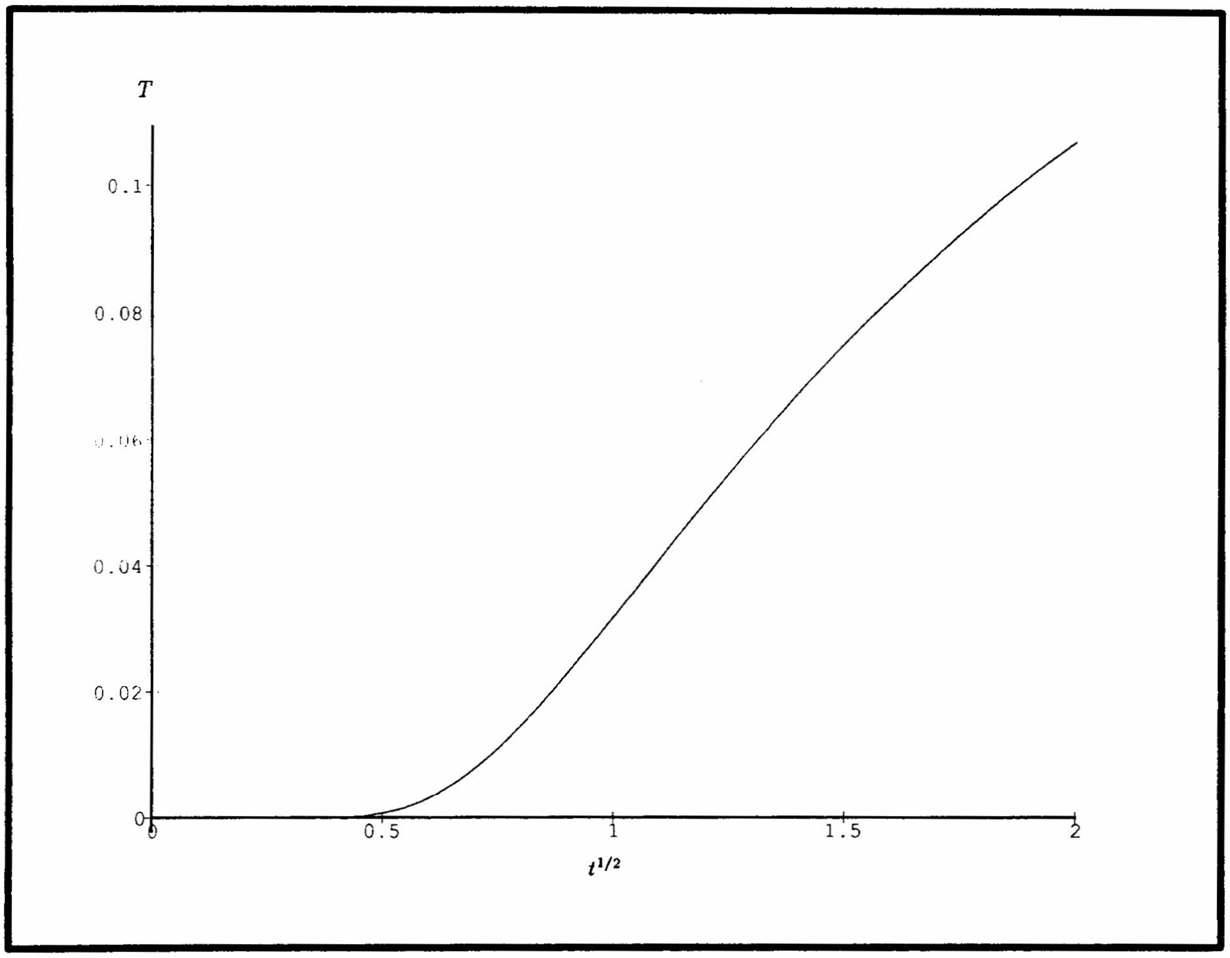

Figure 8. The rise in the temperature of the fish at a distance of 2 below the disc probe over short times.